Large Deviations of nonconventional sums.

S.R.S. Varadhan
Courant Institute, NYU

Florence
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- Joint work with Yuri Kifer

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- What if $F_i = f(X_i, X_{2i})$ or $f(X_i, X_{2i}, \ldots, X_{ki})$ for some k.

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- $$\sum_{r=1}^{\infty} \frac{1}{2^{r+1}} \log E[\exp[\sum_{i=1}^{r} f(X_i, X_{i+1})]]$$

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- Let its size be l_q

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- $p_\ell = \Pi_{i=1}^{m}(1 - \frac{1}{p_i})[\frac{1}{c(\ell)} - \frac{1}{c(\ell+1)}]$

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- $\rho(T)$ is the spectral radius.

$$\psi = \log \rho(T)$$

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- $\psi = \psi(\alpha)$ may exist for nice π

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- Example. $F_n = \sum_{i=1}^{n-m} f(x_i, x_{i+1}, \ldots, x_{i+m})$

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- Let $k \to \infty$
- $\sup_\mu [\mathcal{F}(\mu) - J(\mu)]$

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- We use this repeatedly to evaluate when $F_i = f(X_i, X_{2i})$
- Write

$$\sum_{i=1}^{2^k n} F_i = \sum_{i=1}^{n} F_i + \sum_{j=1}^{k-1} \sum_{i=2^j n+1}^{2^{j+1} n} F_i$$

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- Consider now $H^2(a_1, \ldots, a_{2^{k-2}n})$ given by

$$\log E[\exp[H^1(x_{2^{k-1}n+1}, x_{2^{k-1}+2}, \ldots, x_{2^k n}) + \sum_{i=1}^{2^{k-2}n} f(a_i, x_{2^{k-1}n+2i})]]$$

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