Exactly solvable directed polymers in the KPZ universality class

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Outline

1. KPZ vs EW universality in 1+1 dimensional models

2. Three exactly solvable models in KPZ class: KPZ equation, semi-discrete polymer, log-gamma polymer

3. Specific results for the log-gamma polymer: stationary process, fluctuation exponents, large deviations

Next talk by N. Zygouras: log-gamma polymer and tropical combinatorics.

Collaborators: Márton Balázs (Budapest), Firas Rassoul-Agha (Utah), Jeremy Quastel (Toronto), Nicos Georgiou (Utah), Ivan Corwin (Microsoft/MIT), Neil O'Connell (Warwick), Nikos Zygouras (Warwick), Michael Damron (Indiana)

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Characterized by fluctuation exponents and limit distributions.

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Edwards-Wilkinson (EW)

- time $\sim n$, spatial correlations $\sim n^{1/2}$, fluctuations $\sim n^{1/4}$
- Gaussian limits

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First a brief look at **EW** class through an example.

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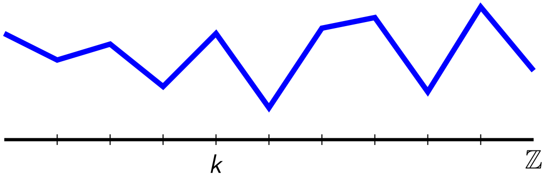

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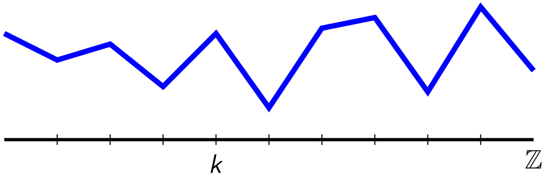

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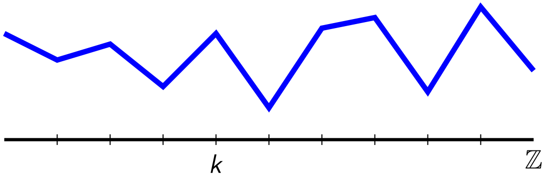

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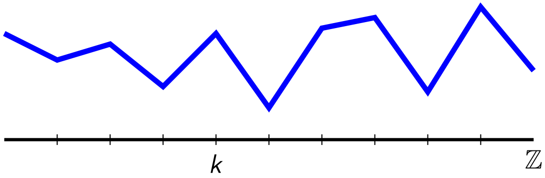

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Model introduced by Ferrari-Fontes EJP 1998.

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Scaled height process

$$z_n(t,r) = n^{-1/4}\big\{\sigma_{\lfloor nt \rfloor}(-\lfloor ntv \rfloor + \lfloor r\sqrt{n}\,\rfloor) - \mu_0 r\sqrt{n}\,\big\}, \quad (t,r) \in \mathbb{R}_+ \times \mathbb{R}.$$

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Theorem. [Balázs, Rassoul-Agha, S. 2006] $z_n(t,r) \Rightarrow Z(t,r)$ where Z is the Gaussian process

$$Z(t,r) \;=\; c_1 \iint\limits_{[0,t]\times\mathbb{R}} p_{\sigma^2(t-s)}(r-x)\,dW(s,x) \;+\; c_2 \int\limits_{\mathbb{R}} p_{\sigma^2 t}(r-x) B(x)\,dx$$

Edwards-Wilkinson (EW) universality

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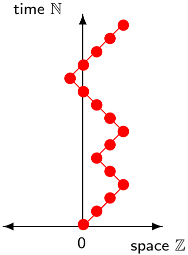
In this class also

- current of independent random walks (incl. RWRE)
- symmetric simple exclusion process
- Hammersley's serial harness process

KPZ class: 1+1 dim directed polymer



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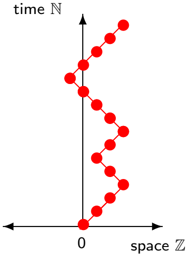

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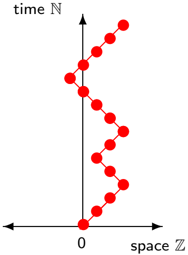

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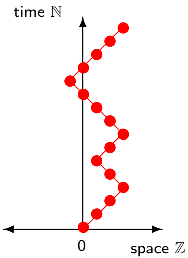

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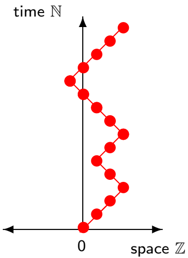

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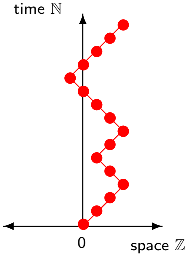

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Model: Huse and Henley 1985

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- Universality close to boundary of lattice.

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Next brief look at KPZ, then focus on log-gamma.

KPZ equation

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$$h_t \ = \ \tfrac{1}{2}\, h_{xx} \, + \, \tfrac{1}{2}\, (h_x)^2 \, + \, \dot{W}$$

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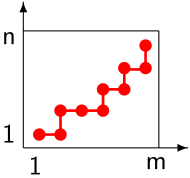
- Probability distribution for $h(t, x)$, narrow wedge initial condition. (Amir-Corwin-Quastel and Sasamoto-Spohn 2011).

Log-gamma polymer



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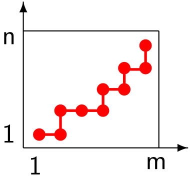

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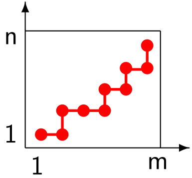

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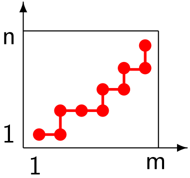

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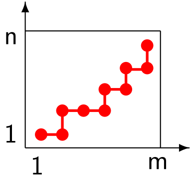

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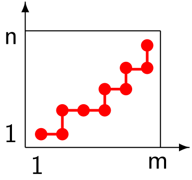

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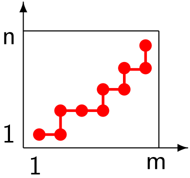

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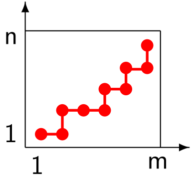

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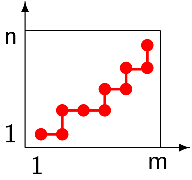

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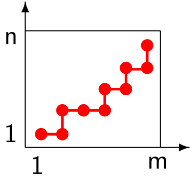

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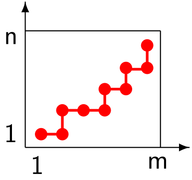

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Fix $0 < \mu < \infty$, take $Y_{i,j}^{-1} \sim \text{Gamma}(\mu)$. **Gamma density:** $f(x) = \frac{1}{\Gamma(\mu)} x^{\mu-1} e^{-x}$

Results:

- Model and KPZ exponents (S 2010).
- Large deviations (Georgiou-S 2011).
- Tropical combinatorics (Corwin-O'Connell-S-Zygouras 2011).
- Tracy-Widom limit (Borodin-Corwin-Remenik 2012).

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As $\beta \to \infty$, polymer measure concentrates on maximizing path(s).

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Limit is the **corner growth model** with Exp weights.

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Next a look at the stationarity and some consequences.

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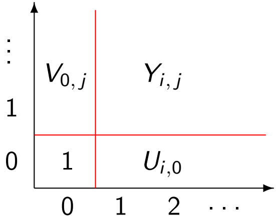

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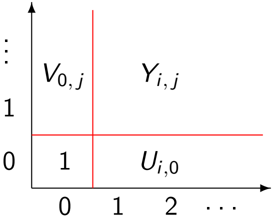

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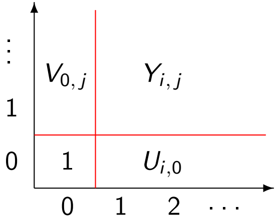

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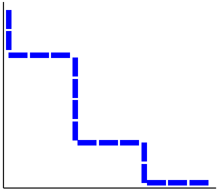

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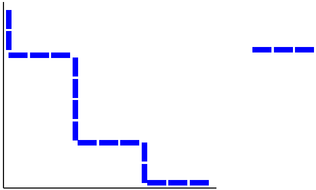

down-right path (z_k) with edges $f_k = \{z_{k-1}, z_k\}$, $k \in \mathbb{Z}$

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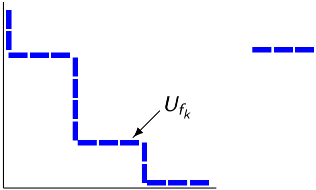

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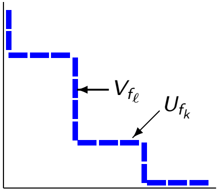

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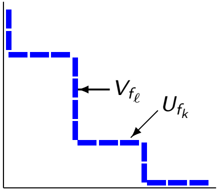
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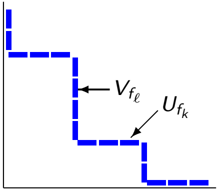
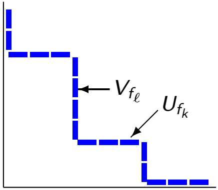


Theorem. For any fixed down-right path, the edge weights $\{U_{f_k}, V_{f_\ell}\}$ along the path are independent, with distributions

$$U_{f_k} \sim \text{Gamma}^{-1}(\theta) \qquad V_{f_\ell} \sim \text{Gamma}^{-1}(\mu - \theta)$$



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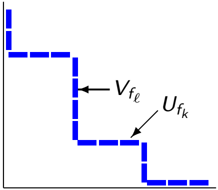

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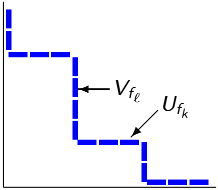

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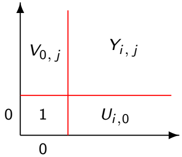
We could call this the **Burke property** of the log-gamma polymer.

Taking advantage of the stationarity



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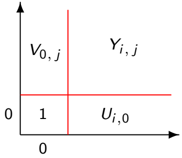

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Coupling of two log-gamma models:

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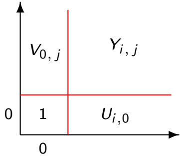

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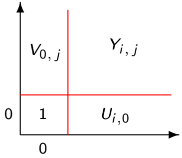

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Let us look at fluctuation exponents for $\log Z$.

Fluctuation exponents: stationary case

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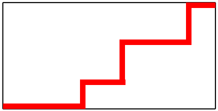

Exit point of path from x-axis

$$\xi_x = \max\{k \geq 0 : x_i = (i, 0) \text{ for } 0 \leq i \leq k\}$$

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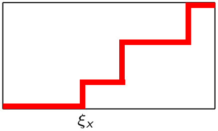

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For $\theta, x > 0$ define positive function

$$L(\theta, x) = \int_0^x \big(\Psi_0(\theta) - \log y\big) x^{-\theta} y^{\theta-1} e^{x-y}\, dy$$

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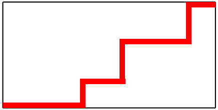

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Theorem. For the stationary case,

$$\mathbb{V}\mathrm{ar}\big[\log Z^{\theta}_{m,n}\big] = n\Psi_1(\mu - \theta) - m\Psi_1(\theta) + 2\, E_{m,n}\Bigg[\sum_{i=1}^{\xi_x} L(\theta, Y^{-1}_{i,0})\Bigg]$$

Remark: polygamma functions

$$\Psi_n(s) = \frac{d^{n+1}}{ds^{n+1}} \log \Gamma(s), \qquad n \geq 0$$

These appear naturally because for $Y^{-1} \sim \text{Gamma}(\mu)$

$$\mathbb{E}(\log Y) = -\Psi_0(\mu) \qquad \text{(digamma function)}$$

$$\mathrm{Var}(\log Y) = \Psi_1(\mu) \qquad \text{(trigamma function)}$$

Fluctuation exponent: stationary case

With $0 < \theta < \mu$ fixed and $N \nearrow \infty$ assume

$$|\, m - N\Psi_1(\mu - \theta)\,| \le CN^{2/3} \quad \text{and} \quad |\, n - N\Psi_1(\theta)\,| \le CN^{2/3} \tag{1}$$

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Theorem: Variance bounds in characteristic direction

For (m, n) as in (1), $\quad C_1 N^{2/3} \le \mathbb{V}\mathrm{ar}(\log Z^{\theta}_{m,n}) \le C_2 N^{2/3}$.

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Theorem: Off-characteristic CLT

Suppose $n = \Psi_1(\theta)N$ and $m = \Psi_1(\mu-\theta)N + \gamma N^{\alpha}$ with $\gamma > 0$, $\alpha > 2/3$. Then

$$N^{-\alpha/2}\Big\{\log Z^{\theta}_{m,n} - \mathbb{E}\big(\log Z^{\theta}_{m,n}\big)\Big\} \Rightarrow \mathcal{N}\big(0, \gamma\Psi_1(\theta)\big)$$

Fluctuation bounds: original i.i.d. case

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$$p_{s,t}(\mu) \equiv \lim_{N\to\infty} \frac{\log Z_{Ns,Nt}}{N} = \inf_{\theta\in(0,\mu)} \{-s\Psi_0(\theta) - t\Psi_0(\mu-\theta)\}$$

Fluctuation bounds: original i.i.d. case

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Remark. Similar bounds exist for path with KPZ exponent $2/3$.

Explicit large deviations for $\log Z$

L.m.g.f. of $\log Y$, $Y \sim \Gamma^{-1}(\mu)$:

$$M_\mu(\xi) = \log \mathbb{E}\big(e^{\xi \log Y}\big) = \begin{cases} \log \Gamma(\mu - \xi) - \log \Gamma(\mu) & \xi \in (-\infty, \mu) \\ \infty & \xi \in [\mu, \infty). \end{cases}$$

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Theorem. [Georgiou, S 2011]

$$\Lambda_{s,t}(\xi) = \begin{cases} p(s,t)\xi & \xi < 0 \\ \inf\limits_{\theta \in (\xi,\mu)} \big\{ tM_\theta(\xi) \; - \; sM_{\mu-\theta}(-\xi) \big\} & 0 \le \xi < \mu \\ \infty & \xi \ge \mu. \end{cases}$$

- $\Lambda_{s,t}$ linear on $\mathbb{R}_-$ because for $r < p(s,t)$

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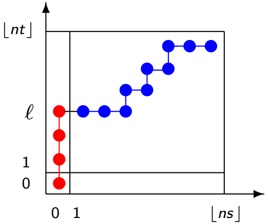
- Proof of formula for $\Lambda_{s,t}$ goes by first finding $J_{s,t}$ and then convex conjugation.

Starting point for proof of large deviations



$$Z^{\theta}_{ns,nt} = \sum_{\ell=1}^{nt} \left(\prod_{j=1}^{\ell} V_{0,j} \right) Z_{(1,\ell),(ns,nt)}$$

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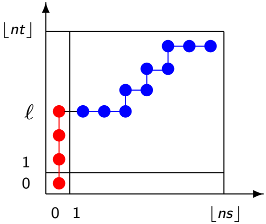

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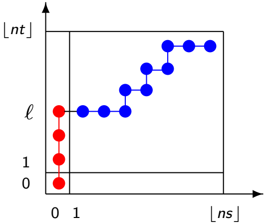

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Now we know LDP for log(l.h.s) and can extract $\log Z$ from the r.h.s.

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Theorem. These exponents valid for stationary semidiscrete polymer. Upper bounds valid for model without boundaries. [Moreno, S, Valkó]

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