Sandpile models on random structures

Wioletta Ruszel

in collaboration with F. Redig (TU Delft) and E. Saada (CNRS and U Paris V)

Technische Universiteit Eindhoven University of Technology

Radboud Universiteit Nijmegen

Motivation

- Abelian Sandpile Model (ASM) was introduced by Bak, Tang and Wiesenfeld ([1]) as paradigmatic example of Self-Organized Criticality (SOC)
- SOC: underlying idea is that simple dynamical mechanism which generates a stationary state in which complex behavior (manifested e.g. through power laws and large avalanches) appears in the limit of large system sizes without any fine-tuning parameter

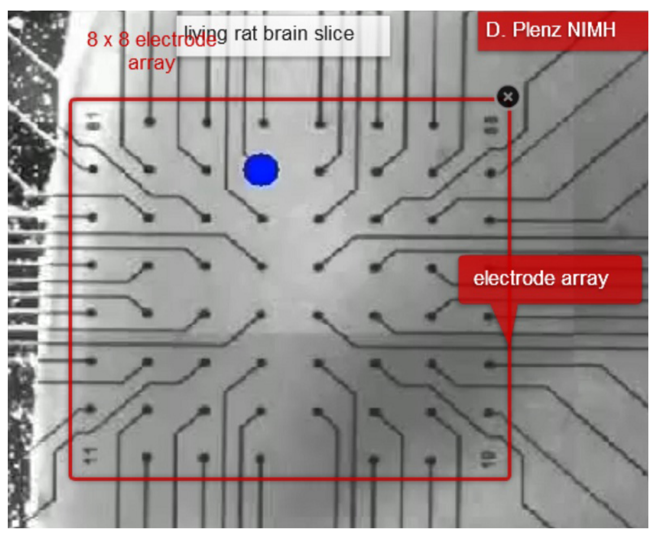


Figure 1: recording of neuronal activity by multi-electrode array, Beggs and Plenz ([2])

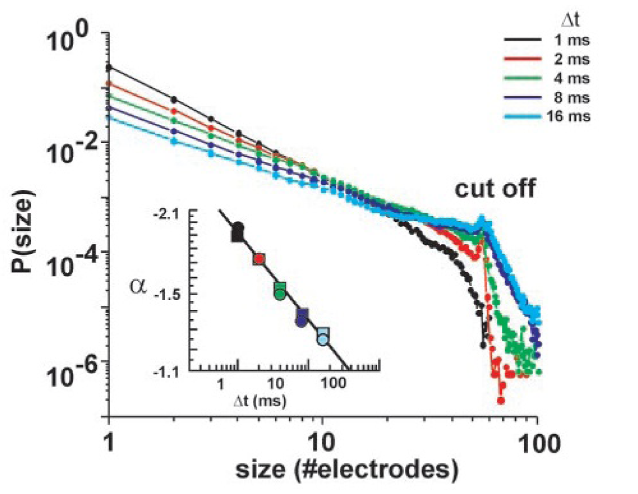


Figure 2: recorded size distributions for avalanches, different time bins

- sandpile dynamics related to neuronal firering
- neuronal connections can be modelled by a random structure (tree or graph)
- power law behaviour of size of neuronal avalanches measured experimentally by Beggs and Plenz ([2])
- we present some results on random binary trees

The model

Model:

- let $\mathscr{T} \subset \mathscr{B}(p)$, finite subtree of the rootless binary tree with branching parameter $p \in [0,1]$ (with probability p each vertex has 2 children and 0 w.p. $1-p$, for $p=1$ we find the Bethe lattice)
- a height configuration η is a map $\eta : \mathscr{T} \to \mathbb{N}^{\mathscr{T}}$,
- η is called stable if $\forall i \in \mathscr{T} : 1 \leq \eta_i \leq 3$
- during a toppling of an unstable site u, the height η_u decreases by 3 and the height of all nearest neighbours of u increases by 1
- $\mathscr{S}(\eta + \delta_u)$ is the unique stable configuration arising from a sequence of topplings upon adding a particle at site u to the stable configuration η, (toppling order does not matter due to the abelian property)
- (commuting) addition operator a_u via $a_u(\eta) := \mathscr{S}(\eta + \delta_u)$

Dynamics:

- dynamics of the sandpile model is the discrete-time Markov chain $\{\eta(n), n \in \mathbb{N}\}$, defined by $\eta(n) = \prod_{i=1}^n a_{X_i}(\eta(0))$ and $(X_i)_{i\in\mathscr{T}}$ are i.i.d. vertices uniformly chosen from $\mathscr{T}$ (at every time step pick a site uniformly at random, add a particle and stabilize if necessary)
- avalanche: $Av(i,\eta)$ denotes the set of sites which have to be toppled upon addition at i in η
- $A_n(i,\mathscr{T})$ denotes the number of connected subsets $\mathscr{C} \subset \mathscr{T}$ containing i and of size n
- $\mu_{\mathscr{T}}$ denotes the unique stationary measure for the dynamics, given $\mathscr{T}$

Example

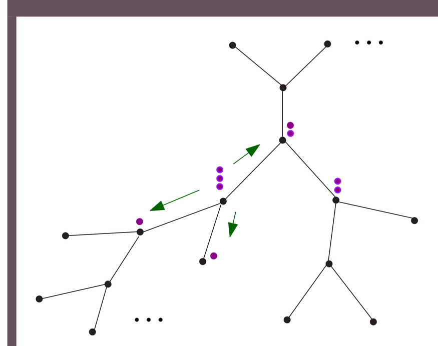


Figure 3: example of an unstable configuration

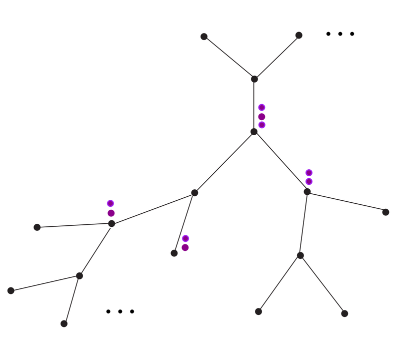


Figure 4: after the first toppling

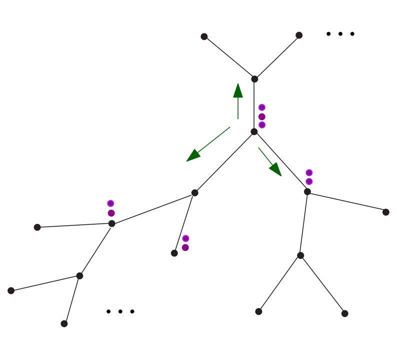
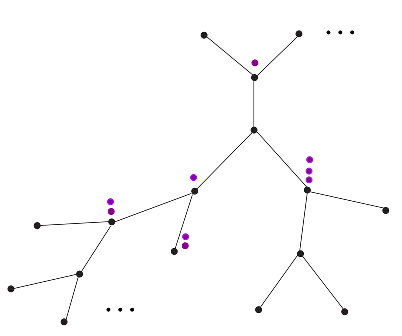
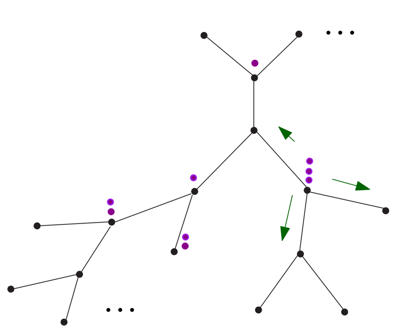


Figure 5: new unstable configuration





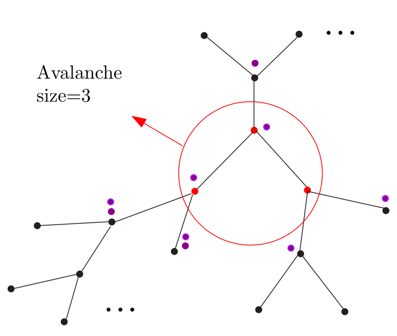


Figure 6: resulting stable configuration, avalanche of size 3

Results

Th. 0.1 ([5]) *Given a realization of the tree $\mathscr{T}$, there exists $C > 0$, such that for all $n \geq 1$,*

$$\mu_{\mathscr{T}}(|Av(o,\eta)| = n) \leq C A_n(o,\mathscr{T}) 4^{-n} 2^{\frac{16}{25}n}. \tag{1}$$

In particular if the tree has growth rate $\limsup_{n\to\infty} \frac{1}{n}\log(A_n(o,\mathscr{T})) < \frac{34}{25}\log(2)$, then the avalanche size decays exponentially.

Th. 0.2 ([5]) *For the stationary binary branching with reproduction p there exists a constant C_2 such that averaged over all trees $\mathscr{T}$,*

$$\mathbb{E}(\mu_{\mathscr{T}}(|Av(o,\eta)| = n)) \leq C_2 2^{\frac{16}{25}n} \left(\frac{p+\sqrt{p}}{2}\right)^n, \tag{2}$$

if $p < p_0 = 0,54511...$ then the averaged probability of an avalanche of size n decays exponentially.

Discussion

- on the lattice $\mathbb{Z}^d$ simulations (see e.g. [7]) suggest the conjecture

$$\lim_{V\to\mathbb{Z}^d} \mu_V(|Av(0,\eta)| > n) \approx Cn^{-\delta} \tag{3}$$

  where μ_V denotes the finite-volume stationary measure
- explicit analysis on the lattice is highly non-trivial, only few rigorous results are known, e.g. about height probabilities and tails of certain correlations functions [4]
- rigorous results only for the Bethe lattice $\mathscr{B}(1)$ by Dhar and Majumdar [3]:
  1. the number of clusters of size n is independent of its shape $A_n(0,\mathscr{B}(1)) = C4^n n^{-3/2}(1+o(1))$
  2. power law decay of avalanche sizes: $\mu_{\mathscr{T}}(|Av(u,\eta)| = n) \approx n^{-3/2}$ where $\mathscr{T} \subset \mathscr{B}(1)$
- we have an exact expression of $\mathbb{E}(A_n(o,\mathscr{T}))$, the dominant contribution comes from the term $((p+\sqrt{p})/2)^n$, in contrast to the Bethe lattice, the (random) number of clusters of size n becomes dependent on the underlying environment
- in [5] we also obtained bounds on the quenched and annealed covariance of height variables at distance n
- combining the results it follows that there is a phase transition phenomenon in the distribution of avalanche sizes: there exists $p_c \in (p_0, 1]$ such that the average avalanche sizes changes from exponential to power law decay!

Current and future investigations

Current:

- avalanches can be decomposed into so-called waves whose size is related to the number of ends of spanning trees
- we investigate quenched and annealed behaviour of avalanche sizes on a Galton-Watson branching process using this correspondance

Future:

- once we understood how sandpile models behave on general random trees, we can extend the analysis to locally tree-like random graphs (e.g. with a power-law degree distribution)
- experimental evidence that functional connectivity of neuronal connections has features of power-law and small-world random graphs (see [6])

References

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