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# Microstructures in one dimensional systems

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Microstructures thus appear on scales very small in macroscopic units yet very large microscopically.

**An example.**

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There is a good theory of the phenomenon in the context of variational problems for suitable energy or free energy functionals.

Stefan Müller and his school among the main contributors and I borrow an example from his lecures on elasticity:

# Müller example.

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Both requests can be simultaneously met and the inf of $F(u)$ is 0:

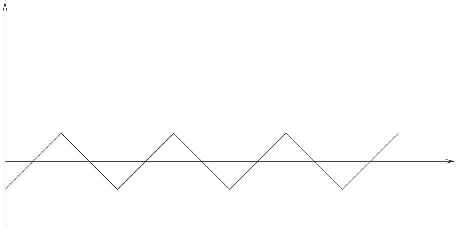

Figure: minimizing sequence: values between $\pm\varepsilon$ with slope $\pm 1$

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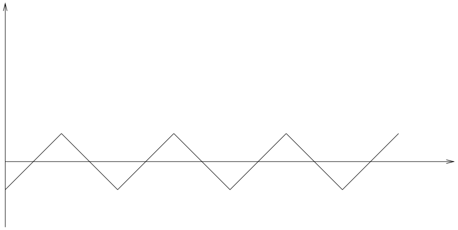

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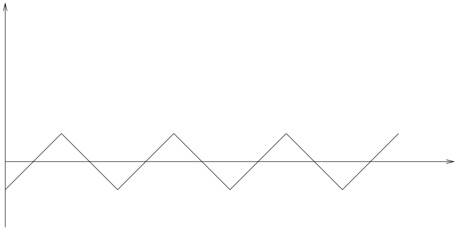

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The integral of $u^2$ is on "scale 1", $du/dx$ on an infinitesimal scale, the two scales are infinitely far apart
and as a consequence the period of oscillations is infinitesimal.

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The role of the two slopes in statistical mechanics is played by the two values of the density at phase transition.

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If we put in the box $\Lambda$ a mass $\rho|\Lambda|$ of fluid with $\rho \in (\rho', \rho'')$, ("canonical constraint") we do not see a homogeneous density $\rho$, but a density $\rho'$ in a subset $\Lambda' \subset \Lambda$ and $\rho''$ in the complement.

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The free energy when the total magnetization is $m$ is:

$$f_L(m) := \frac{1}{2L} \inf \Big\{ F(u) \Bigm| \frac{1}{2L} \int_{-L}^{L} u = m \Big\}$$

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Take $u(x) = -1$ for $x < -\varepsilon$, $u(x) = 1$ for $x > \varepsilon$ and

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In most of the space we see magnetization $\pm 1$.

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We want this local canonical constraint to arise from the action of long range interactions.

Going back to Müller example, $\int u^2$ plays the role of the long range forces and the $\pm 1$ slopes correspond to the two pure phase densities $\rho'$ and $\rho''$.

**Kac potentials**

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The limit free energy in Kac systems is:

$$f_\beta^{\mathrm{m.f.}}(\rho) = \phi_\beta^{**}(\rho), \quad \phi_\beta(\rho) = e(\rho) + f_\beta^{\mathrm{s.r.}}(\rho)$$

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$$H_\gamma(q) = \int_{\mathbb{R}^d} e(J_\gamma * q), \;\; J_\gamma * q(r) = \sum_{q_i \in q} J_\gamma(r, q_i)$$

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The free energy of $H_\gamma + H^{\mathrm{s.r.}}$ in the limit $\gamma \to 0$ is $f_\beta^{\mathrm{m.f.}}$.

**Examples of Kac potentials.**

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$e(\rho) = -\frac{\rho^2}{2} + \frac{\rho^4}{4!}$, $H^{\rm s.r.} = 0$.
It has phase transitions when $\gamma > 0$ is small

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We want:

- $f_\beta^{\rm s.r.}(\rho)$ linear (i.e. with a phase transition) in $(\rho', \rho'')$
- $\phi_\beta^{**}(\rho) = \phi_\beta(\rho)$ in $(\rho', \rho'')$

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Put in a cube $\Lambda \subset \mathbb{R}^3$ of large side $L$ a mass $\rho|\Lambda|$ and observe the system by partitioning $\Lambda$ into cubes $C^{(\ell)}$ of side $\ell \ll L$.
We observe different behaviors when varying $\rho$.

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For small densities $\rho$:

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- This is uniform no matter how large is $L$
- Accuracy improves with $\ell$: less and less probable to be in the unlikely event, the fraction of bad cubes decreases, in the good cubes closeness to $\rho$ improves.

**Phase transition.**

The gas phase persists till $\rho \leq \rho^-_{\beta,\gamma;R}$.
In the interval $(\rho^-_{\beta,\gamma;R}, \rho^+_{\beta,\gamma;R})$ there is a phase transition.
Take $\rho \in (\rho^-_{\beta,\gamma;R}, \rho^+_{\beta,\gamma;R})$, then

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E. Pulvirenti, D. Tsagkarogiannis and EP, in preparation.

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What happens at larger densities is only conjectures.

The pure hard spheres system in $d = 3$ has a phase transition (from disorder to order) with forbidden density interval $(\rho', \rho'')$ (as suggested by numerical computations).
The liquid phase persists till $\rho \leq \rho'$ while the system is in its solid phase when $\rho \geq \rho''$.
For $\rho \in (\rho', \rho'')$ there are microstructures.

Disregarding as usual unlikely fluctuations we see:

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- Second regime. $\ell > \gamma^a$, $a > 1$, then in a large fraction of cubes $C^{(\ell)}$ the density is close to $\rho$.
- The picture is uniform in $L$. The second regime gets sharper as $\ell$ increases while to have sharper accuracy in the first regime we must take $\gamma$ smaller.

# Rigorous results.

Microstructures have been found as minimizers of free energy functionals in anelastic crystals and micromagnetism.
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Periodic minimizers in the $d = 1$ case with computation of period.
Bounds in $d > 1$.

**One dimensional Ising at $T > 0$.**
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$$f_\beta^{\mathrm{m.f.}}(m) = \phi_\beta^{**}(m),\ \phi_\beta(m) = -\frac{m^2}{2} - \frac{1}{\beta} S(m)$$

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For $\beta > 1$: $\phi_\beta^{**}(m) < \phi_\beta(m)$ for $|m| < m_\beta$,
$m_\beta = \tanh\{\beta m_\beta\} > 0$.

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The competition here is between the Kac energy which plays the role of the short range forces while the long range forces are due to the entropy which in $d = 1$ is so strong to prevent phase transitions.

# DLR measures

By equivalence of ensembles we can reduce to study the typical configurations of DLR measures with magnetic field $h$ (and $\beta > 1$). Relevant $h$ are those for which the average spin $m$ varies in $(-m_\beta, m_\beta)$.

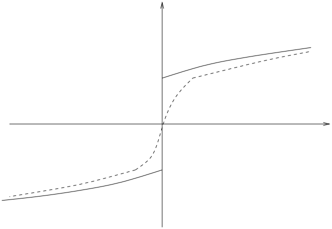

Figure: $m$ versus $h$ in mean field and $\gamma > 0$ (dashed line)

Turns out that: $h \approx e^{-c\gamma^{-1}}$.

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Too many information in $\sigma$, we just want to know where empirical magnetization is close to $\pm m_\beta$.

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Too many information in $\sigma$, we just want to know where empirical magnetization is close to $\pm m_\beta$.

Introduce a phase indicator: $\Theta(x;\sigma)$: $\Theta(x;\sigma) = \pm 1$ means that empirical averages around $x$ are close to $\pm m_\beta$; otherwise $\Theta(x;\sigma) = 0$ no phase can be recognized at $x$.

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Definition such that intervals with $\Theta = 1$ and $\Theta = -1$ are separated by $\Theta = 0$.

**Interfaces**

$\sigma$ gives rise to a sequence $(\ell_i, \ell'_i)$, $i \in \mathbb{Z}$: $\ell_i$ length of $i$-th interval with $\Theta \geq 0$; $\ell'_i$ length of the successive interval with $\Theta \leq 0$.

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However we can cluster together the intervals in a suitable way so that the clustered intervals have the law of a renewal process.