Internal Diffusion Limited Aggregation, from the symmetric to the asymmetric case

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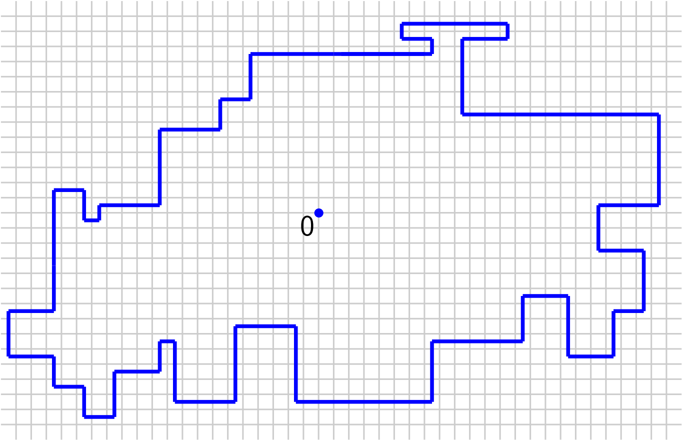
August 28th, 2012

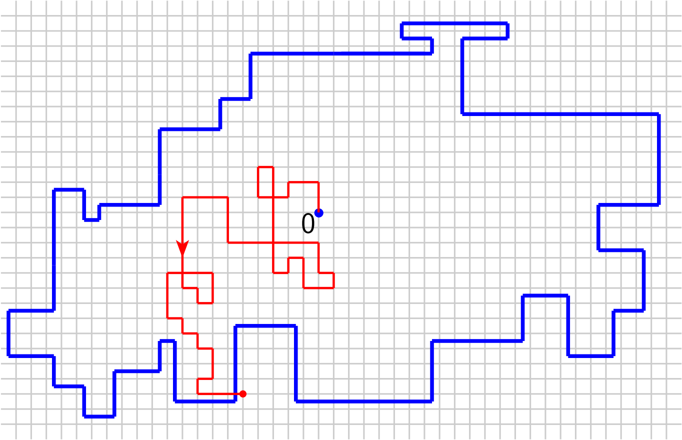
Diffusion rule

Start with any finite set of $\mathbb{Z}^d$ containing the origin ($\mathbb{Z}^2$ on the figure below).



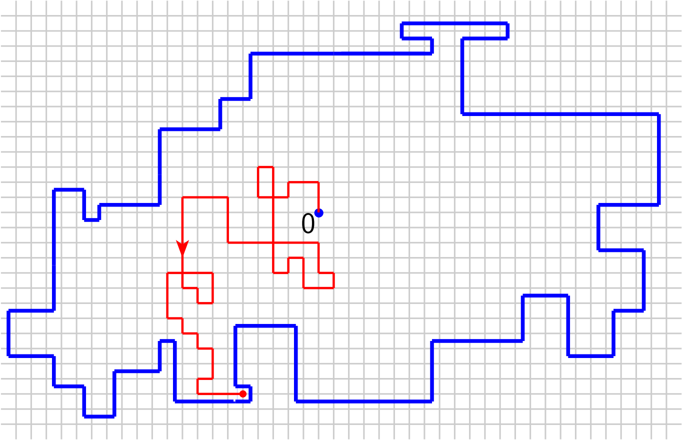
Diffusion rule

Launch a random walk from 0 and let it evolve until it hits the border of the cluster.



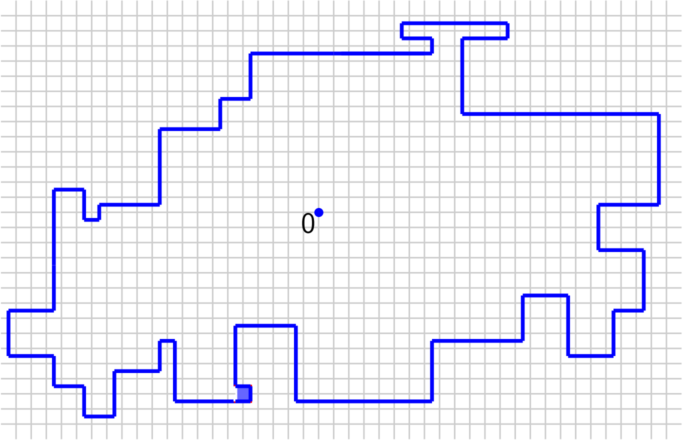
Diffusion rule

Add the first point visited by the random walk outside the cluster to the cluster.



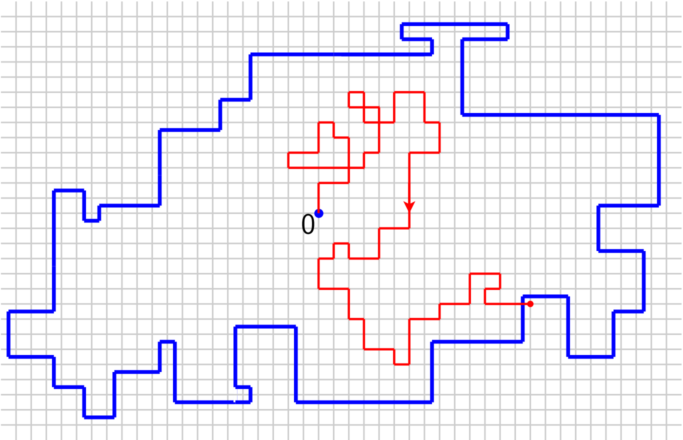
Diffusion rule

This defines the cluster at time $n + 1$.



Diffusion rule

The aggregate is built recursively, by launching independant random walks from the origin.



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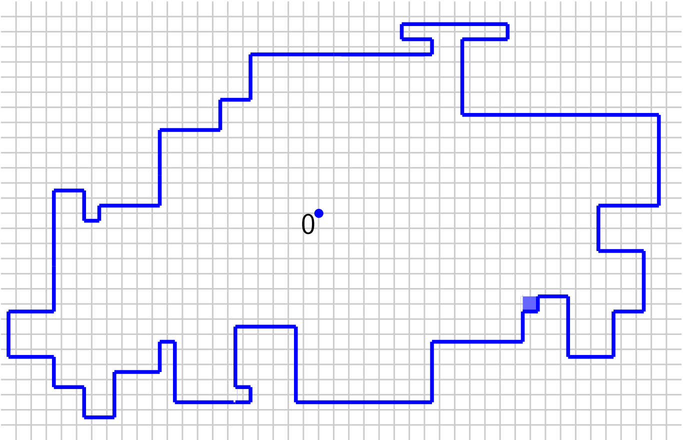
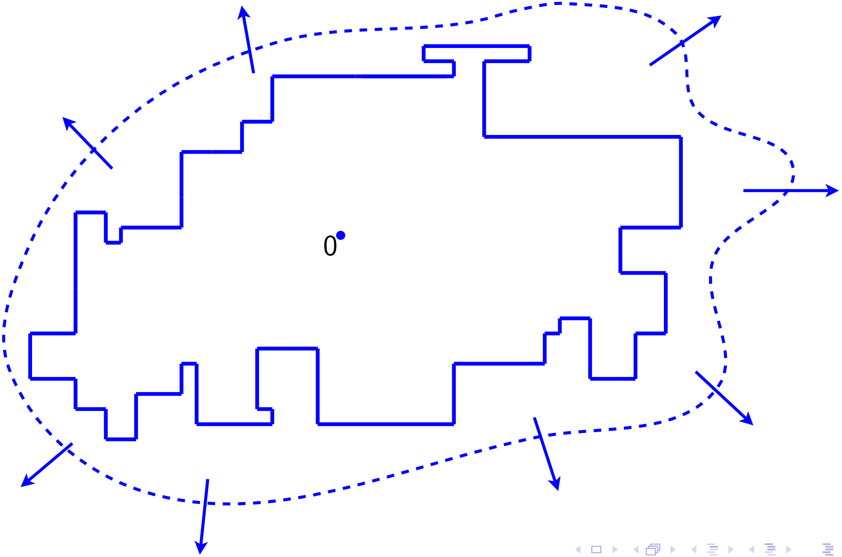


Diffusion rule

As n tends to ∞, the cluster at time n grows.



Definitions

For a first look at the model, we make the following asumptions :

- We will work on the lattice $\mathbb{Z}^d$, $d \geq 2$.
- Our random walks will be simple random walks on this lattice.
- Let us start at time 1 with the cluster $A(1) = \{0\}$.

Hausdorff Convergence

Theorem

(Lawler, Bramson and Griffeath, 1992) *Let* ω_d *be the volume of the euclidian unit ball of* $\mathbb{R}^d$.

Then the cluster $A(\omega_d n^d)$ *for the simple random walk normalized by* n *converges to the euclidian unit ball with respect to the Hausdorff distance.*

Simulation

The odometer function

A key to understanding the mechanics of the iDLA model is the odometer function, introduced by Levine and Peres.
It can be defined for all $z \in \mathbb{Z}^d$ as follows :

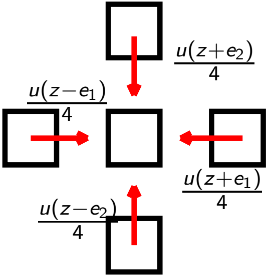
$$u_n(z) := \sum_{i=1}^{\omega_d n^d} \sum_{j=1}^{\sigma_i} \mathbf{1}_{S^i(j)=z},$$

where σ_i is the time at which the i-th random walk adds to the cluster.
This function measures the total number of particles that passed through z on their way to add to the cluster, with repetitions for multiple passages of the same particle.

Mass exchanges and the odometer function

Since the odometer function counts the total number of walks that go trough a given vertex z, we expect that about $\frac{1}{2d}$ of these walks will go to a given neighbour of z. Hence the relations for the total mass received in or emmited from z :

Total mass received in z :



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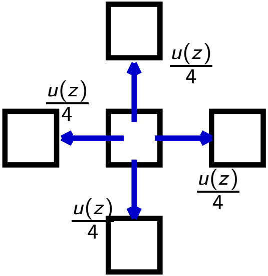

Local relation

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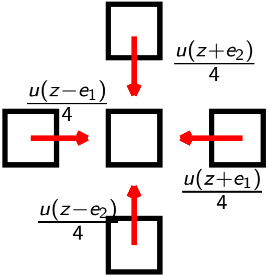

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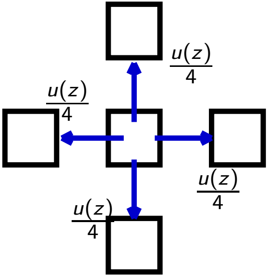
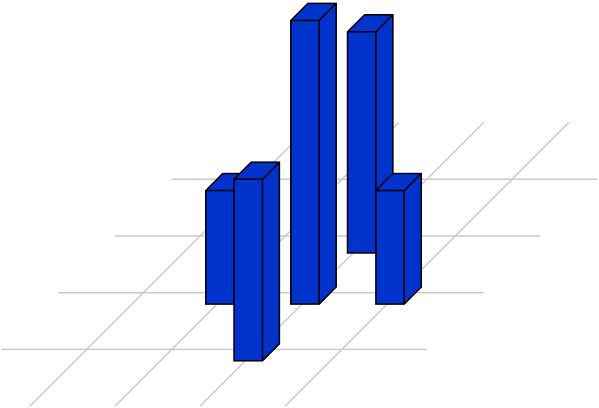


Let ν be the final mass repartition, and σ be the initial mass repartition in the model. Then we expect the local relation :

$$\mathbb{E}\left[\frac{1}{4}\sum_{y\sim z} u(y) - u(z)\right] \sim \nu(z) - \sigma(z)$$

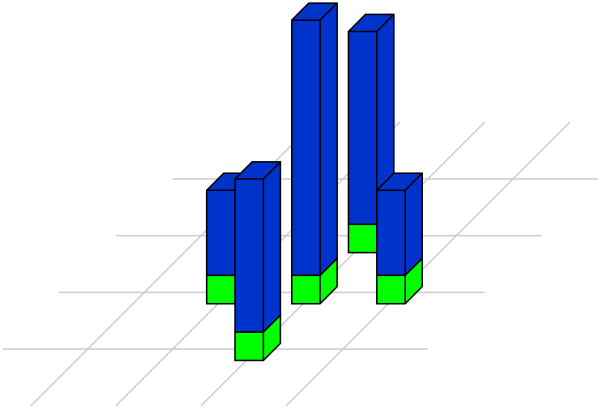
The Divisible Sandpile model

- On $\mathbb{Z}^d$, we start with a compactly supported mass distribution (a pile of sand).



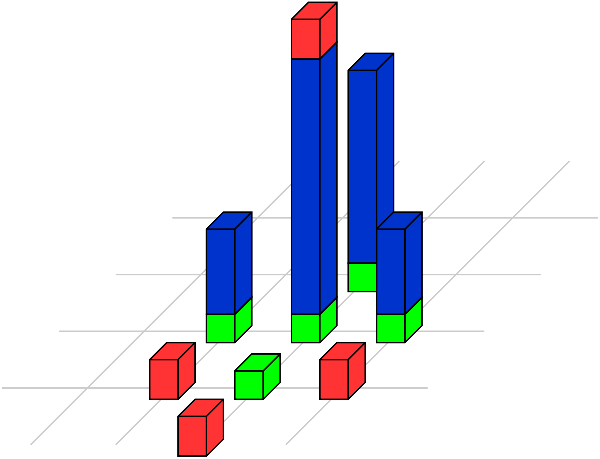
The Divisible Sandpile model

- Each vertex can safely hold mass 1 of sand. When it holds more, it can topple.



The Divisible Sandpile model

- When a site topples, the excess is split equally between its nearest neighbours.



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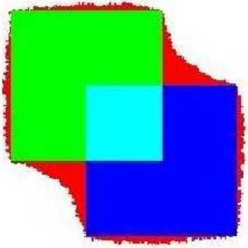
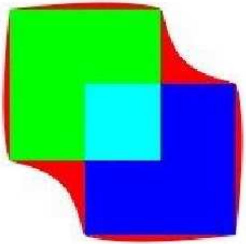
- When all topplings realised, the mass distribution converges to a final state.
- This final state is made of aggregates of full vertices surrounded by vertices with mass between 0 and 1.
- The final state does not depend on the order of topplings.

Initial mass repartition

These models do not depend on the order of the walks (Markov) or the topplings (Abelian property), so that we can run them with any initial mass distribution.
In the following example, an iDLA and a divisible sandpile model are run starting from a configuration with one particle in each vertex of the squares (mass 1), except in the intersection, where there are initially two particles (mass 2).





Convergence Theorem

Theorem (Levine, Peres, 2008)

Under suitable conditions for the initial mass repartition, when the mesh tends to 0*, the sets obtained as the results of : :*

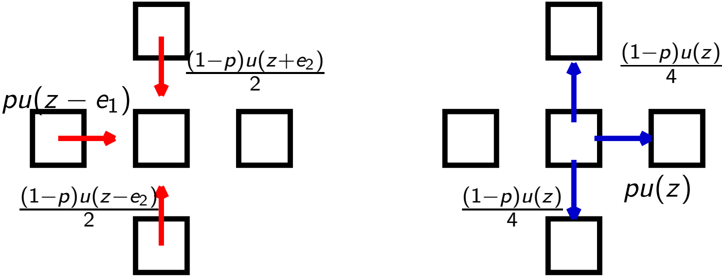
- *the divisible sandpile model,*
- *the iDLA model*

converge to the same limiting shape (with respect to the Hausdorff measure, and with probability one in the iDLA case).

Drifted Random Walks

We build the iDLA cluster with drifted random walks :
$P\left(S(t+1)-S(t)=\pm e_i\right)=\frac{1-p}{2(d-1)}$ for $i=1\cdots d-1$, and
$P\left(S(t+1)-S(t)=\ e_d\right)=p.$
The local mass equation becomes :

Total mass received in z : Total mass emmited from z :



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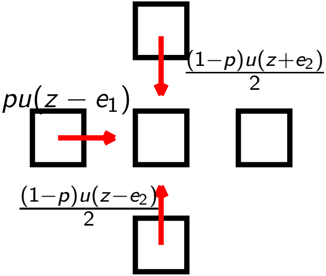

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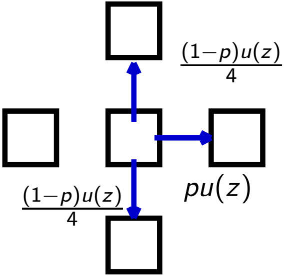

Let ν be the final mass repartition, and σ the initial mass repartition. Then we expect :

$$\frac{1-p}{2}\left(\sum_{y\sim z,(y-z)\in e_1^{\perp}} u(y)-u(z)\right)+p\left(u(z-e_1)-u(z)\right)\approx \nu(z)-\sigma(z)$$

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- The last coordinate is normalized by $n^{\frac{2}{d+1}}$ and called the time coordinate.

Convergence Theorem

Theorem

Let $\mathcal{A}_n$ be the normalized drifted iDLA aggregate. Then, almost surely, $\mathcal{A}_n$ converges to D, where $D \subset \mathbb{R}^{d-1} \times \mathbb{R}_+$ has the following property :

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$$\frac{1-p}{2(d-1)}\Delta\phi + p\frac{\partial\phi}{\partial t} = 0,$$

Then ϕ has the following mean value property :

$$\int_D \phi(z,t)d(z,t) = |D|\phi(0).$$

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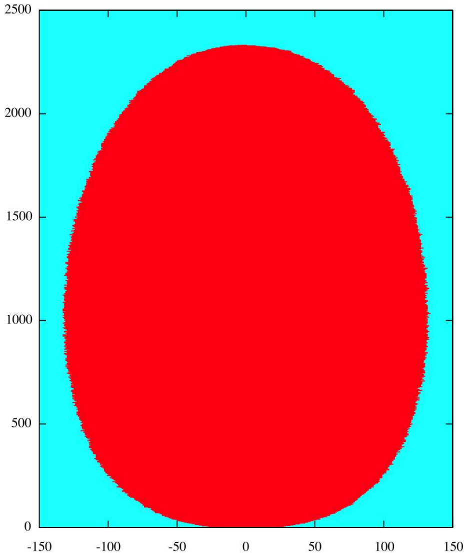
Moreover, D is bounded in time and space directions.



The aggregate with 500 000 particles.

The “unfair divisible sandpile” model

We define the unfair divisible sandpile model in the same way as the classical divisible sandpile, with the difference that the mass excess is now split according to the step distribution of S.

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- The mass configuration verifies the local equation

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Discrete parabolic free boundary problem

We define the discrete caloric operator

$$\mathcal{K}f(x) = (1-p)\tilde{\Delta}f(x) - p\left(f(x) - f(x-e_1)\right),$$

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$$\mathcal{K}u_n(x) = \begin{cases} 1-n & \text{at the origin} \\ 1 & \text{inside the aggregate} \\ 0 & \text{at distance} \geq 2 \text{ from the aggregate.} \end{cases}$$

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Then $u_n + \gamma_n$ is the least super-caloric majorant of γ_n.

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\begin{tikzcd}
\gamma_n \arrow[rr, "\text{least discrete super-caloric majorant}", "\textit{discret}"'] \arrow[d, "\text{converges to}"'] & & u_n + \gamma_n \arrow[d, "?"] \\
\gamma \arrow[rr, "\text{least continuous super-caloric majorant}", "\textit{continu}"'] & & u + \gamma
\end{tikzcd}
$$

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$$\begin{array}{ccc}
\gamma_n & \xrightarrow[\textit{discret}]{\text{least discrete super-caloric majorant}} & u_n + \gamma_n \\
{\scriptstyle \text{converges to}}\big\downarrow & & \big\downarrow{\scriptstyle \text{Yes !}} \\
\gamma & \xrightarrow[\textit{continu}]{\text{least continuous super-caloric majorant}} & u + \gamma
\end{array}$$

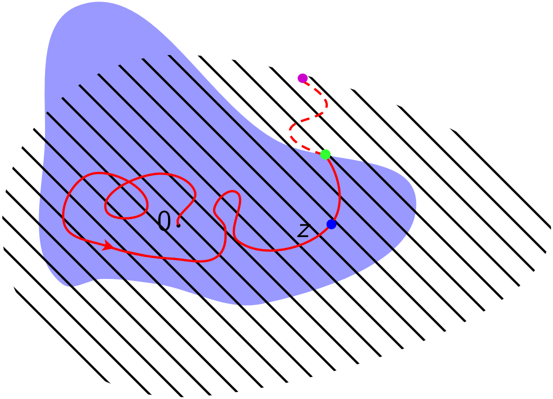
Moreover, u verifies the PDE :

$$\frac{1-p}{2(d-1)}\Delta u - p\frac{\partial u}{\partial t} = \mathbf{1}_{u>0} - \delta_0.$$

Back to iDLA

Define :

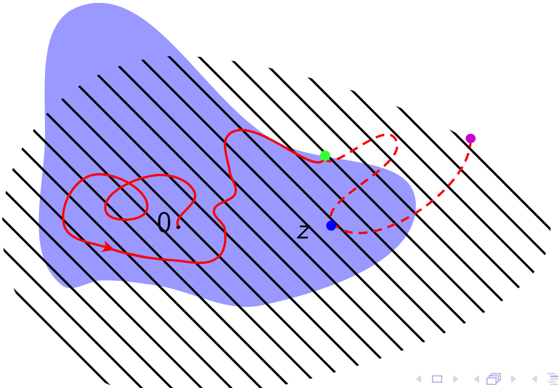
- N the number of walks that hit z before leaving D_n or adding to the aggregate.



Back to iDLA

Define :

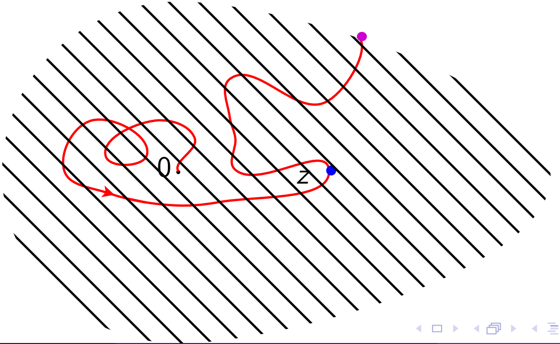
- N the number of walks that hit z before leaving D_n or adding to the aggregate.
- L the number of walks that hit z before leaving D_n but after adding to the aggregate.



Back to iDLA

Define :

- N the number of walks that hit z before leaving D_n or adding to the aggregate.
- L the number of walks that hit z before leaving D_n but after adding to the aggregate.
- M the sum of these variables ; it is the number of walks that hit z before exiting D_n.



Reindexing the walks :

$$
\begin{aligned}
L_n &= \sum_{i=1}^{n} \mathbf{1}_{\nu_i < \tau_z^i < \tau_{D_n}^i} \\
&\leq \sum_{y \in D_n} \mathbf{1}_{\tau_z^y < \tau_{D_n}^y} = \tilde{L}_n,
\end{aligned}
$$

Define :

$$f_{n,D_n}(z) = g_{n,D_n}(z,z)\mathbb{E}\left(M_n(z) - \tilde{L}_n(z)\right).$$

$$\begin{aligned} f_{n,D_n}(z) &= g_{n,D_n}(z,z)\left(\sum_{i=1}^{n}\mathbb{P}\left(\tau_z^i < \tau_{D_n}^i\right) - \sum_{y\in D_n}\mathbb{P}\left(\tau_z^y < \tau_{D_n}^y\right)\right) \\ &= g_{n,D_n}(z,z)\sum_{y\in D_n}(\delta_0(y)n - 1)\mathbb{P}(\tau_z^y < \tau_{D_n}^y). \\ &= \sum_{y\in D_n}(\delta_0(y)n - 1)g_{n,D_n}(y,z) \end{aligned}$$

So that $f_{n,D_n}(z)$ and u_n satisfy the same discrete PDE inside D_n. Hence a control on $\mathbb{E}\left(M_n(z) - \tilde{L}_n(z)\right)$.