



Metastability of zero range processes

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J. Beltrán, P. Ferrari, A. Gaudilli re, V. Sisko, C. Landim





Condensation in zero range processes

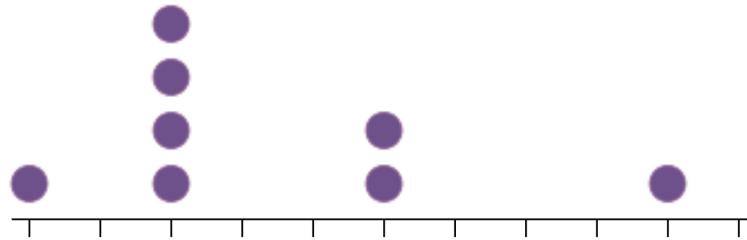
- $\mathbb{T}_L = \{1, \dots, L\}$
- State space $\mathbb{N}^{\mathbb{T}_L}$
- configurations $\eta = \{\eta_x : x \in \mathbb{T}_L\}$





Dynamics

- $g : \mathbb{N} \rightarrow \mathbb{R}_+$ $g(0) = 0$ $g(k) > 0$ $0 < p \leq 1$
- $x \rightarrow x + 1$ at rate $pg(\eta_x)$ $x \rightarrow x - 1$ at rate $(1 - p)g(\eta_x)$
- $g(k) = k$ independent random walks
- $g(k) = \mathbf{1}\{k \geq 1\}$ queues and servers
- $g \downarrow$ sticky





Canonical stationary states

- N number of particles
- $E_{L,N} = \{\eta \in \mathbb{N}^{\mathbb{T}_L} : \sum_{x \in \mathbb{T}_L} \eta_x = N\}$
- $\{\eta(t) : t \geq 0\}$ irreducible
- Exists unique stationary state $\mu_{L,N}$





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Equivalence of ensembles:

- Cylinder function $f \quad f = f(\eta_{-m}, \dots, \eta_m)$
- $\lim_{\substack{L \rightarrow \infty \\ N/L \rightarrow \rho}} E_{\mu_{L,N}}[f] = E_{\nu_\rho}[f]$
- $\mathbb{N}^{\mathbb{Z}}$ stationary state (Grand canonical)
- Number of particles conserved, $\{\nu_\rho : \rho \geq 0\} \quad E_{\nu_\rho}[\eta_0] = \rho$





Grand canonical stationary states

- Partition function: $Z(\varphi) = \sum_{k \geq 0} \frac{\varphi^k}{g(k)!}, \quad \varphi \geq 0$
- $g(0)! = 1, \quad g(k)! = g(1) \cdots g(k)$
- $g(1) = 1 \quad g(k) = \left(\frac{k}{k-1}\right)^\alpha \quad k \geq 2 \quad \alpha > 0 \quad g(k)! = k^\alpha$
- $\varphi^* < \infty$ radius of convergence of $Z \quad \varphi^* = 1$
- $\varphi < \varphi^*$ $\hat{\nu}_\varphi$ product measure on $\mathbb{N}^\mathbb{Z}$
- $\hat{\nu}_\varphi\{\eta : \eta_x = k\} = \frac{1}{Z(\varphi)} \frac{\varphi^k}{g(k)!}$





Equivalence of ensembles

- $\hat{\nu}_\varphi \{ \eta : \eta_x = k \} = \frac{1}{Z(\varphi)} \frac{\varphi^k}{g(k)!} \quad 0 \leq \varphi < \varphi^*$
- $R(\varphi) = E_{\hat{\nu}_\varphi} [\eta_0] = \frac{1}{Z(\varphi)} \sum_{k \geq 1} k \frac{\varphi^k}{g(k)!} = \frac{\varphi Z'(\varphi)}{Z(\varphi)} = \varphi \frac{d}{d\varphi} \log Z(\varphi)$
- $R(0) = 0 \quad R$ strictly increasing
- $\rho^* = \lim_{\varphi \rightarrow \varphi^*} R(\varphi) \quad R : [0, \varphi^*) \rightarrow [0, \rho^*) \quad \Phi = R^{-1}$
- $0 \leq \rho < \rho^* \quad \nu_\rho = \hat{\nu}_{\Phi(\rho)}$
- $E_{\nu_\rho} [\eta_0] = E_{\hat{\nu}_{\Phi(\rho)}} [\eta_0] = R(\Phi(\rho)) = \rho$
- Cylinder function $f \quad \rho < \rho^* \quad \lim_{\substack{L \rightarrow \infty \\ N/L \rightarrow \rho}} E_{\mu_{L,N}} [f] = E_{\nu_\rho} [f]$
- Local central limit theorem (Kipnis - L)





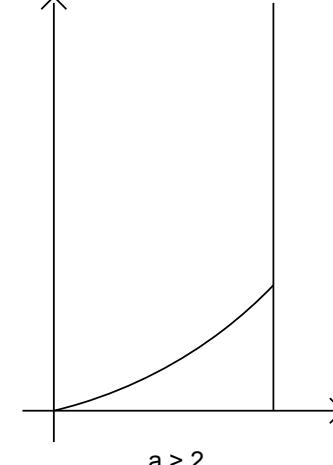
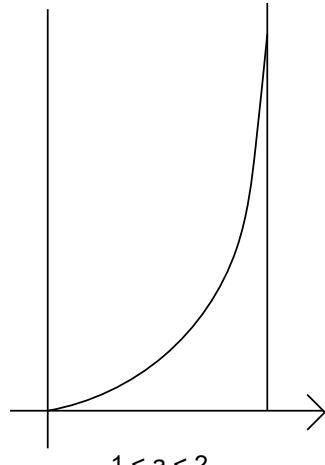
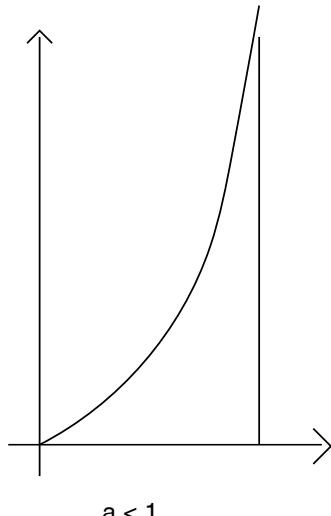
Critical density

- $\rho^* = \lim_{\varphi \rightarrow \varphi^*} R(\varphi) = \lim_{\varphi \rightarrow \varphi^*} \varphi \frac{d}{d\varphi} \log Z(\varphi) = \varphi^* \lim_{\varphi \rightarrow \varphi^*} \frac{d}{d\varphi} \log Z(\varphi)$

- $g(1) = 1 \quad g(k) = \left(\frac{k}{k-1}\right)^\alpha \quad k \geq 2 \quad \alpha > 0 \quad g(k)! = k^\alpha \quad \text{sticky}$

- $Z(\varphi) = \sum_{k \geq 0} \frac{\varphi^k}{g(k)!} \quad R(\varphi) = \frac{1}{Z(\varphi)} \sum_{k \geq 0} k \frac{\varphi^k}{g(k)!} \quad \varphi^* = 1$

- $\alpha \leq 1 \quad 1 < \alpha \leq 2 \quad \alpha > 2$





Phase transition

- $\alpha \leq 1 \quad Z(\varphi^*) = \infty \quad \rho^* = \infty$
- $1 < \alpha \leq 2 \quad Z(\varphi^*) < \infty \quad \rho^* = \infty$
- $\alpha > 2 \quad Z(\varphi^*) < \infty \quad \rho^* < \infty$
- **Problem:** $\mu_{L,N}$ if $N/L = \rho > \rho^*$?





Condensation

- Evans, Godrèche, Grosskinsky, Schuetz, Spohn. Bose–Einstein condensation.
- Grosskinsky, Schuetz, Spohn (JSP 2003)
- Armendariz, Loulakis (PTRF 2009)
- T removes the site with largest number of particles
- $\{N_L : L \geq 1\} \quad N_L/L \rightarrow \rho > \rho^* \quad \mu_{L,N} T^{-1} \sim \nu_{\rho^*}$
- Ferrari, Sisko, L. (JSP 2008) Beltrán, L. (JSP 2010)
- $\alpha > 1$ L fixed
- $1 \ll \ell_N \ll N \quad \lim_{N \rightarrow \infty} \mu_{L,N} \{\max_{1 \leq x \leq L} \eta_x \geq N - \ell_N\} = 1$
- Let $N \uparrow \infty$, $\mu_{L,N} T^{-1} \rightarrow \nu_{\rho^*}$





Evolution of the condensate

- L fixed $N \uparrow \infty$
- Zero-range process $\{\eta(t) : t \geq 0\}$ on $E_{L,N}$
- Suppose $\eta_1(0) = N$

Questions:

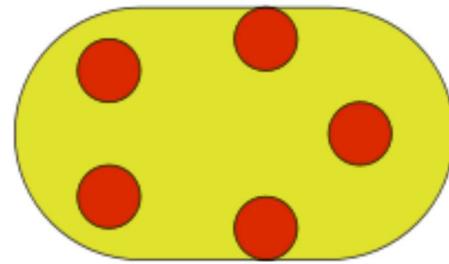
- $T_N = \inf\{t > 0 : \max\{\eta_x(t) : x \neq 0\} = N\}$
- Order of T_N ?
- $Y_1 = x$ if $\eta_x(T_N) = N$
- Distribution of Y_1 ? Nearest-neighbor, Uniform?





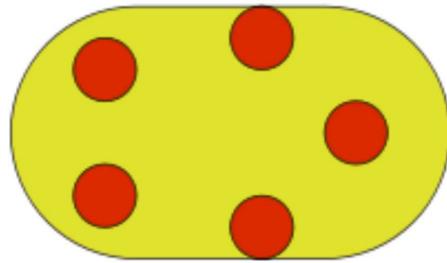
Metastability

- Fix $1 \ll \ell_N \ll N$
- $\mathcal{E}_N^x = \{\eta : \eta(x) \geq N - \ell_N\} \quad 1 \leq x \leq L$
- $\mathcal{E}_N = \bigcup_{x=1}^L \mathcal{E}_N^x \quad E_{L,N} = \mathcal{E}_N \cup \Delta_N$





Metastability



- (M1) Starting from \mathcal{E}_N^x , the process thermalizes on \mathcal{E}_N^x before leaving this set.
- (M2) On an appropriate time scale, process jumps from \mathcal{E}_N^x to \mathcal{E}_N^y at exponential times.
- (M3) On that time scale, the time spent on Δ_N is negligible.



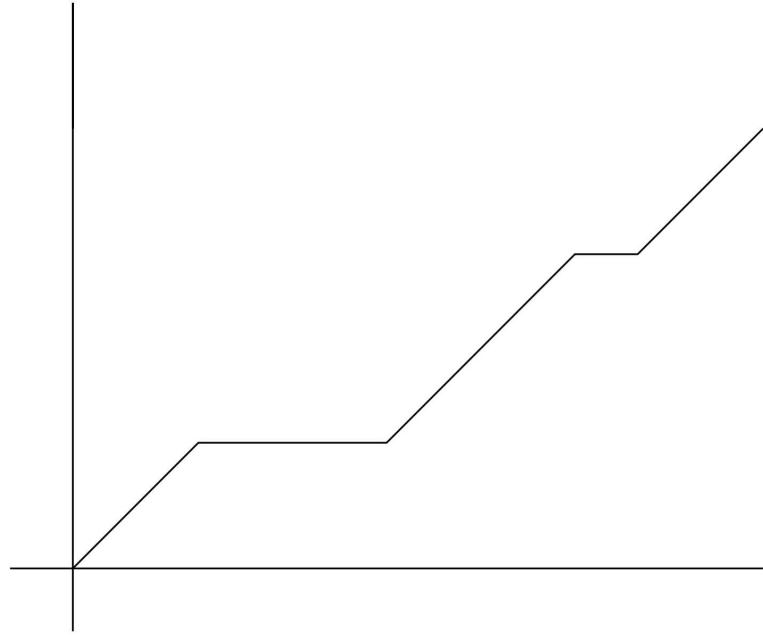
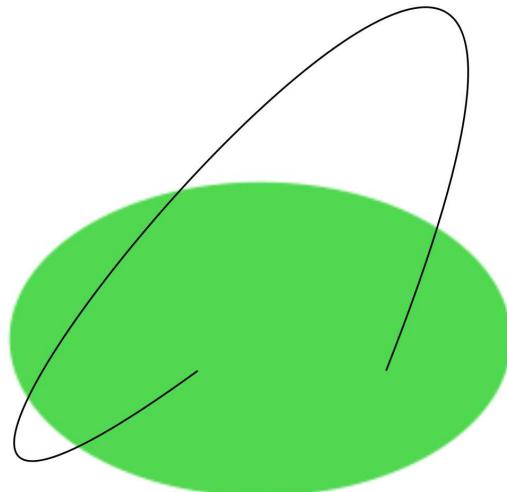


A martingale approach to Metastability



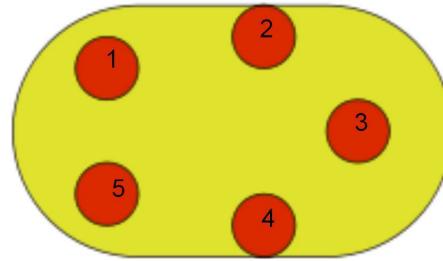
Trace

- $\eta^{\mathcal{E}_N}(t)$ trace of $\{\eta(t) : t \geq 0\}$ on $\mathcal{E}_N = \bigcup_{x=1}^L \mathcal{E}_N^x$:
- $T(t) = \int_0^t \mathbf{1}\{\eta(s) \in \mathcal{E}_N\} ds$
- $S(t) = \sup\{s : T(s) \leq t\}$
- $\eta^{\mathcal{E}_N}(t) = \eta(S(t))$ Markov process on \mathcal{E}_N



Asymptotic Markovian Dynamics

- $\eta^{\mathcal{E}_N}(t)$ trace of $\{\eta(t) : t \geq 0\}$ on $\mathcal{E}_N = \bigcup_{x=1}^L \mathcal{E}_N^x$
- $\Psi_N : \mathcal{E}_N \rightarrow \{1, \dots, L\}$ $\Psi_N(\eta) = x$ iff $\eta \in \mathcal{E}_N^x$
- $X_N(t) = \Psi_N(\eta^{\mathcal{E}_N}(t))$ not Markovian



(M2): $X_N(t\theta_N) \rightarrow X(t)$ Markov process on $\{1, \dots, L\}$.



Martingale approach

- $X_t^N = \Psi(\eta^{\mathcal{E}}(t\theta_N)) \longrightarrow X_t$
- Tightness X_t^N
- X_t solves martingale problem $F : \{1, \dots, L\} \rightarrow \mathbb{R}$
 - $F(X_t) - F(X_0) - \int_0^t (\mathcal{L}F)(X_s) ds$
 - $F(X_t) - F(X_0) - \int_0^t \sum_{y=1}^L r(X_s, y)[F(y) - F(X_s)] ds$





Martingale approach

- $F(X_t) - F(X_0) - \int_0^t \sum_{y=1}^L r(X_s, y)[F(y) - F(X_s)] ds$

- $M_t^N = F(X_t^N) - F(\Psi(\eta^{\mathcal{E}}(0))) - \int_0^{t\theta_N} [\textcolor{red}{L}_{\mathcal{E}}(F \circ \Psi)](\eta^{\mathcal{E}}(s)) ds$

$$\begin{aligned}
 [L_{\mathcal{E}}(F \circ \Psi)](\eta) &= \sum_{\xi \in \mathcal{E}_N} \textcolor{red}{R}^{\mathcal{E}_N}(\eta, \xi) \{(F \circ \Psi)(\xi) - (F \circ \Psi)(\eta)\} \\
 &= \sum_{x,y=1}^L [F(y) - F(x)] \sum_{\xi \in \mathcal{E}_N^y} R^{\mathcal{E}_N}(\eta, \xi) \mathbf{1}\{\eta \in \mathcal{E}_N^x\} \\
 &= \sum_{x,y=1}^L [F(y) - F(x)] R^{\mathcal{E}_N}(\eta, \mathcal{E}_N^y) \mathbf{1}\{\eta \in \mathcal{E}_N^x\}
 \end{aligned}$$





Metastable ergodicity

- $F(X_t^N) - F(X_0^N) - \sum_{x,y=1}^L [F(y) - F(x)] \int_0^{t\theta_N} R^{\mathcal{E}_N}(\eta(s), \mathcal{E}_N^y) \mathbf{1}\{\eta(s) \in \mathcal{E}_N^x\} ds$
- $G_{x,y}(\eta) = R^{\mathcal{E}_N}(\eta, \mathcal{E}_N^y) \mathbf{1}\{\eta \in \mathcal{E}_N^x\}$
- $\int_0^{t\theta_N} G_{x,y}(\eta(s)) ds$
- $\mathcal{P} = \sigma\{\mathcal{E}_N^x : 1 \leq x \leq L\} \quad \hat{G}_{x,y} = E_{\mu_N}[G_{x,y}(\eta)|\mathcal{P}]$

$$\int_0^{t\theta_N} \left\{ G_{x,y}(\eta_s^{\mathcal{E}}) - \hat{G}_{x,y}(\eta_s^{\mathcal{E}}) \right\} ds \longrightarrow 0 \tag{C1}$$

- $\hat{G}_{x,y}(\eta) = \frac{1}{\mu_N(\mathcal{E}_N^x)} \sum_{\eta \in \mathcal{E}_N^x} \mu_N(\eta) R^{\mathcal{E}_N}(\eta, \mathcal{E}_N^y) =: r_{\mathcal{E}_N}(x, y)$
- $F(X_t^N) - F(X_0^N) - \int_0^t \sum_{y=1}^L \theta_N r_{\mathcal{E}_N}(X_{s\theta_N}^N, y) [F(y) - F(X_{s\theta_N}^N)] ds$





Asymptotic behavior of rates

- $F(X_t^N) - F(X_0^N) - \int_0^t \sum_{y=1}^L \theta_N r_{\mathcal{E}_N}(X_{s\theta_N}^N, y) [F(y) - F(X_{s\theta_N}^N)] ds$

- $r_{\mathcal{E}_N}(x, y) = \frac{1}{\mu_N(\mathcal{E}_N^x)} \sum_{\eta \in \mathcal{E}_N^x} \mu_N(\eta) R^{\mathcal{E}_N}(\eta, \mathcal{E}_N^y)$

$$\theta_N r_{\mathcal{E}_N}(x, y) \longrightarrow r(x, y) \quad (\mathbf{C2})$$

$$\lim_{N \rightarrow \infty} \sup_{\eta \in \mathcal{E}_N} \mathbb{E}_\eta^N \left[\int_0^t \mathbf{1}\{\eta(s\theta_N) \in \Delta_N\} ds \right] = 0 \quad (\mathbf{C3})$$

- Nothing is said if $\eta(0) \in \Delta_N$.





Martingale approach

- Th (Beltrán, L.): Sufficient conditions for a ergodic Markov process on a countable space to be metastable.
- All conditions are expressed in terms of the measure $\mu_{L,N}$ and capacities.





Potential Theory, Capacity

- Markov process $\{\eta(t) : t \geq 0\}$ on E
- Rates $R(\eta, \xi)$ $\lambda(\eta) = \sum_{\xi \neq \eta} R(\eta, \xi)$ $M(\eta) = \mu(\eta) \lambda(\eta)$
- Hitting and return times

$$H_A = \inf\{t > 0 ; \eta(t) \in A\}$$

$$H_A^+ = \inf\{t > 0 ; \eta(t) \in A \ \exists s < t \ \eta(s) \neq \eta(0)\}$$

- Capacity $A, B \subset E, A \cap B = \emptyset$

$$\text{cap}(A, B) = \sum_{\eta \in A} M(\eta) \mathbb{P}_\eta[H_B^+ < H_A^+]$$





Dirichlet principle

- Generator L , Dirichlet form $D(f) = \langle (-L)f, f \rangle_\mu$





Dirichlet principle

- Generator L , Dirichlet form $D(f) = \langle (-L)f, f \rangle_\mu$
- Reversible
- $\text{cap}(A, B) = \inf_F \langle F, (-L)F \rangle_\mu = D(V_{A,B})$
- $F_A = 1, F_B = 0$
- $V_{A,B}(\eta) = \mathbb{P}_\eta[H_A < H_B].$





Dirichlet principle

- Generator L , Dirichlet form $D(f) = \langle (-L)f, f \rangle_\mu$
- **Reversible**
 - $\text{cap}(A, B) = \inf_F \langle F, (-L)F \rangle_\mu = D(V_{A,B})$
 - $F_A = 1, F_B = 0$
 - $V_{A,B}(\eta) = \mathbb{P}_\eta[H_A < H_B].$
- **Non-reversible** (Pinsky, Doyle, Gaudilli re-L.)
 - $\text{cap}(A, B) = \inf_F \sup_H \left\{ 2\langle F, LH \rangle_\mu - \langle H, (-L)H \rangle_\mu \right\},$
 - $F_A = 1, F_B = 0 \quad H_A = C_1, H_B = 0$
 - $F_{A,B} = (1/2)\{V_{A,B} + V_{A,B}^*\} \quad H_{A,B} = V_{A,B}$





Process visits points, Condition (C1)

- $\forall x \quad \exists \xi^x \in \mathcal{E}_N^x$

$$\lim_{N \rightarrow \infty} \inf_{\eta \in \mathcal{E}_N^x} \mathbb{P}_{\eta}^N \left[H(\xi^x) < H(\cup_{y \neq x} \mathcal{E}_N^y) \right] = 1.$$





Process visits points, Condition (C1)

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- $T_{\text{mix}}^x \ll \theta_N \quad T_{\text{rel}}^x \ll \theta_N$





Process visits points, Condition (C1)

- $\forall x \quad \exists \xi^x \in \mathcal{E}_N^x$

$$\lim_{N \rightarrow \infty} \inf_{\eta \in \mathcal{E}_N^x} \mathbb{P}_\eta^N \left[H(\xi^x) < H(\cup_{y \neq x} \mathcal{E}_N^y) \right] = 1.$$

- $T_{\text{mix}}^x \ll \theta_N \quad T_{\text{rel}}^x \ll \theta_N$

$$\lim_{N \rightarrow \infty} \sup_{\eta \in \mathcal{E}_N^x} \frac{\text{cap}_N(\mathcal{E}_N^x, \cup_{y \neq x} \mathcal{E}_N^y)}{\text{cap}_N(\eta, \xi^x)} = 0$$





Capacity and mean rates, Condition (C2)

$$\bullet \quad r_{\mathcal{E}_N}(x, y) = \frac{1}{\mu_N(\mathcal{E}_N^x)} \sum_{\eta \in \mathcal{E}_N^x} \mu_N(\eta) R^{\mathcal{E}_N}(\eta, \mathcal{E}_N^y)$$

$$\theta_N r_{\mathcal{E}_N}(x, y) \longrightarrow r(x, y) \quad (\mathbf{C2})$$

$$A \subset K \subset E \quad \mu(A) r_K(A, K \setminus A) = \text{cap}(A, K \setminus A)$$





Capacity and mean rates, Condition (C2)

- Reversible: $A, B \subset K \subset E \quad A \cap B = \emptyset$

$$2\mu(A) r_K(A, B) = \text{cap}(A, K \setminus A) + \text{cap}(B, K \setminus B) - \text{cap}(A \cup B, K \setminus [A \cup B])$$





Capacity and mean rates, Condition (C2)

- Reversible: $A, B \subset K \subset E \quad A \cap B = \emptyset$

$$2\mu(A)r_K(A, B) = \text{cap}(A, K \setminus A) + \text{cap}(B, K \setminus B) - \text{cap}(A \cup B, K \setminus [A \cup B])$$

- Non-reversible: $A, B \subset K \subset E \quad A \cap B = \emptyset$

$$\inf_F \sup_H \left\{ 2\langle F, LH \rangle_\mu - \langle H, (-L)H \rangle_\mu \right\}$$

- $F_A = 1, F_B = C_1, F_{K \setminus (A \cup B)} = 0 \quad H_A = C_2, H_B = C_3,$
 $H_{K \setminus (A \cup B)} = 0$

- $F^{\text{opt}} \quad H^{\text{opt}}$

$$H_B^{\text{opt}} = \frac{r_K(B, A)}{r_K(B, K \setminus B)}$$





Condition (C3)

- Assume asymptotic process has no absorbing points. For all x :

$$\lim_{N \rightarrow \infty} \frac{\mu_N(\Delta_N)}{\mu_N(\mathcal{E}_N^x)} = 0$$





Theorem 1

$$\lim_{N \rightarrow \infty} \sup_{\eta \in \mathcal{E}_N^x} \frac{\text{cap}\left(\mathcal{E}_N^x, \bigcup_{y \neq x} \mathcal{E}_N^y\right)}{\text{cap}(\eta, \xi^{x,N})} = 0 \quad (\text{H1})$$

- $R_N^{\mathcal{E}_N}$ rates of the trace process on \mathcal{E}_N

$$r_{\mathcal{E}_N}(x, y) = \frac{1}{\mu_N(\mathcal{E}_N^x)} \sum_{\eta \in \mathcal{E}_N^x} \mu_N(\eta) R_N^{\mathcal{E}_N}(\eta, \mathcal{E}_N^y)$$

$$\lim_{N \rightarrow \infty} \theta_N r_{\mathcal{E}_N}(x, y) = r(x, y) \quad (\text{H2})$$

- Assume that process with rates r has no absorbing points

$$\lim_{N \rightarrow \infty} \frac{\mu_N(\Delta_N)}{\mu_N(\mathcal{E}_N^x)} = 0 \quad (\text{H3})$$





Theorem 2: Reversible zero range process

- $\alpha > 1 \quad \theta_N = N^{1+\alpha}$
- $\text{cap}_S(x, y)$ capacity of the random walk on \mathbb{T}_L
- $\ell_N^{1+\alpha(L-1)}/N^{1+\alpha} \rightarrow 0$
- Conditions (H1), (H2), (H3) are in force.

$$\lim_{N \rightarrow \infty} N^{1+\alpha} r_{\mathcal{E}_N}(x, y) = C(\alpha) L \text{cap}_S(x, y)$$



Theorem 3: Totally asymmetric zero range process

- $\alpha > 3$
- $\text{cap}_S(x, y) = L^{-1}$
- $\ell_N^{1+\alpha(L-1)}/N^{1+\alpha} \rightarrow 0$
- Conditions (H1), (H2), (H3) are in force.

$$\lim_{N \rightarrow \infty} N^{1+\alpha} r_{\mathcal{E}_N}(x, y) = C(\alpha) L \text{cap}_S(x, y)$$



Convergence of finite dimensional distributions

$$\max_{x \in S} T_x^{\text{rel}} \ll \theta_N \quad (\text{L1})$$

$$\lim_{N \rightarrow \infty} \theta_N r_N(x, y) = r(x, y), \quad x \neq y \in S \quad (\text{L2})$$

• $\nu_N(\mathcal{E}_N^{x_0}) = 1 \quad \mathcal{M}_x = \min \{ \pi_{\mathcal{E}}(\mathcal{E}_N^x), 1 - \pi_{\mathcal{E}}(\mathcal{E}_N^x) \}$

$$E_{\pi_{\mathcal{E}}} \left[\left(\frac{d\nu_N}{d\pi_{\mathcal{E}}} \right)^2 \right] \leq \frac{C_0}{\max_{x \in S} \mathcal{M}_x}$$





Convergence of finite dimensional distributions

$$\max_{x \in S} T_x^{\text{rel}} \ll \theta_N \quad (\text{L1})$$

$$\lim_{N \rightarrow \infty} \theta_N r_N(x, y) = r(x, y), \quad x \neq y \in S \quad (\text{L2})$$

• $\nu_N(\mathcal{E}_N^{x_0}) = 1 \quad \mathcal{M}_x = \min \{ \pi_{\mathcal{E}}(\mathcal{E}_N^x), 1 - \pi_{\mathcal{E}}(\mathcal{E}_N^x) \}$

$$E_{\pi_{\mathcal{E}}} \left[\left(\frac{d\pi_{x_0}}{d\pi_{\mathcal{E}}} \right)^2 \right] = \frac{1}{\pi_{\mathcal{E}}(\mathcal{E}_N^{x_0})} \leq \frac{C_0}{\max_{x \in S} \mathcal{M}_x}$$

$$T_{\mathbf{r}, x_0}^{\text{mix}} \ll \gamma^{-1} \ll \theta_N \quad \lim_{N \rightarrow \infty} \mathbb{P}_{\nu_N}^{\mathcal{E}} [H_{\mathcal{E}_N^{x_0}} \leq \gamma^{-1}] = 0. \quad (\text{L3})$$





Convergence

$$\max_{x \in S} T_x^{\text{rel}} \ll \theta_N \quad (\text{L1})$$

$$\lim_{N \rightarrow \infty} \theta_N r_N(x, y) = r(x, y), \quad x \neq y \in S \quad (\text{L2})$$

• $\nu_N(\mathcal{E}_N^{x_0}) = 1 \quad \mathcal{M}_x = \min \{ \pi_{\mathcal{E}}(\mathcal{E}_N^x), 1 - \pi_{\mathcal{E}}(\mathcal{E}_N^x) \}$

$$E_{\pi_{\mathcal{E}}} \left[\left(\frac{d\pi_{x_0}}{d\pi_{\mathcal{E}}} \right)^2 \right] = \frac{1}{\pi_{\mathcal{E}}(\mathcal{E}_N^{x_0})} \leq \frac{C_0}{\max_{x \in S} \mathcal{M}_x}$$

$$T_{\mathbf{r}, x_0}^{\text{mix}} \ll \gamma^{-1} \ll \theta_N \quad \lim_{N \rightarrow \infty} \mathbb{P}_{\nu_N}^{\mathcal{E}} [H_{\mathcal{E}_N^{x_0}} \leq \gamma^{-1}] = 0. \quad (\text{L3})$$

$$T_{\mathbf{r}, x_0}^{\text{mix}} \ll \gamma^{-1} \ll \theta_N \quad \lim_{N \rightarrow \infty} \sup_{\eta \in \mathcal{E}_N^{x_0}} \mathbb{P}_{\eta}^{\mathcal{E}} [H_{\mathcal{E}_N^{x_0}} \leq \gamma^{-1}] = 0. \quad (\text{L3U})$$





Comments

- Result holds for all reversible cases
- $L = L_N$? Nucleation phase
- To compute $\text{cap}_N\left(\bigcup_{x \in A} \mathcal{E}_N^x, \bigcup_{y \notin A} \mathcal{E}_N^y\right)$
- $\text{cap}_N(A, B) = \inf \{D_N(f) : f_A = 1, f_B = 0\}$

Lower bound:

- Advantage: disregards unimportant bonds in DF
- Difficulty: Uniform over all functions

Upper bound:

- Advantage: Estimate DF one candidate
- Difficulty: Have to estimate all bonds

