A Contact Process with Competitive Immigration on $\mathbb{Z}^d$

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Motivation

Introduced in the '50s by Drs. E. Knipling and R. Bushland [6], the **Sterile Insect Technique** (SIT) involves mass production of a target species, sterilisation and releasing into the wild on a sustained basis and in sufficient numbers to achieve appropriate overflooding ratios. Sterile males find and mate with fertile females without viable offspring, hence the target population is reduced. It is the only environmentally-friendly, low-cost and accurate technology available that involves action against the total population over a period of several generations.

1. Description of the model

The contact process with immigration $(\eta_t)_{t\geq 0}$ is a Markov process of state space $F^{\mathbb{Z}^d}$ with $F = \{0, 1, 2, 3\}$:
$\forall x \in \mathbb{Z}^d$, $\eta_t(x) = 0$ means that at time t site x is "vacant", $\eta_t(x) = 1$ (resp. 2) that x is "occupied" by individuals of type 1 (resp. 2) and $\eta_t(x) = 3$ that x is occupied by both species 1 and 2.
For the dynamics, let $n_i(x, \eta) = \text{card}\{y \in \mathbb{Z}^d : \|y - x\|_2 = 1\}$ be the number of neighbors of site x in state $i \in \{1, 3\}$. Transition rates at x are :

$$\begin{array}{llll} 0 \to 1 & \text{at rate } \lambda_1 n_1(x,\eta) + \lambda_2 n_3(x,\eta) & 1 \to 0 & \text{at rate } 1 \\ 0 \to 2 & \text{at rate } r & 2 \to 0 & \text{at rate } 1 \\ 1 \to 3 & \text{at rate } r & 3 \to 2 & \text{at rate } 1 \\ & & 3 \to 1 & \text{at rate } 1 \end{array}$$

On sites in state 3, there is a competition between species, so that the birth rate of the 1-type species should be strictly smaller on sites in state 3 than on sites in state 1 : we assume $\lambda_2 < \lambda_1$.

2. What if $r = 0$?

If $r = 0$, our model is equivalent to a basic contact process $(\xi_t)_{t\geq 0}$. This Markov process on $\{0, 1\}^{\mathbb{Z}^d}$, introduced by T.E. Harris [3], has the transition rates at site x :

$$\begin{array}{ll} 0 \to 1 & \text{at rate } \lambda_1 n_1(x,\eta) \\ 1 \to 0 & \text{at rate } 1 \end{array}$$

For $(\xi_t)_{t\geq 0}$, there exists a critical birth rate $\lambda_c \in (0, \infty)$ such that

$$\mathbb{P}(\xi_t^0 \neq \emptyset,\ \forall t \geq 0) = 0, \quad \text{if } \lambda \leq \lambda_c \qquad \text{(extinction)},$$
$$\mathbb{P}(\xi_t^0 \neq \emptyset,\ \forall t \geq 0) > 0, \quad \text{if } \lambda > \lambda_c \qquad \text{(survival)},$$

where ξ_t^0 is such that $\xi_0^0(x) = \mathbf{1}_{\{0\}}(x)$.
We refer to [4, 5] by T.M. Liggett and [1, 2] by R. Durrett for details.

3. Objectives

Our interest lies in the population of 1s : What is the influence of the parameter of immigration r on the behaviour of the 1-type species ? Does there exist a critical value of this parameter which might involve a phase transition ?
We study the existence, uniqueness and some bounds of this critical value.

4. Main results

Study of a phase transition according to the values of r

Let $A_t^0 = \{x \in \mathbb{Z}^d : \eta_t(x) \in \{1, 3\} \mid \eta_0(x) = \mathbf{1}_{\{0\}}(x)\}$ and $d \geq 2$.

Proposition 1 The process $(\eta_t)_{t\geq 0}$ is monotone in r. The survival probability of 1s is a non-increasing function in r.

Proposition 2 **Criteria of survival and extinction.**
• There exists $r_0 \in (0, \infty)$ such that for all $r \leq r_0$ the 1-type individuals survive with probability strictly positive.
• There exists $r_1 \in (0, \infty)$ such that for all $r \geq r_1$ the 1-type individuals die out almost surely.

We define the critical value we are interested in by

$$r_c = \inf\{r \geq 0 : \mathbb{P}_r(A_t^0 > 0 \text{ for all } t \geq 0) > 0\}$$

Proposition 3 The critical contact process with immigration dies out :

$$\mathbb{P}_{r_c}(\exists\, t \geq 0\ A_t = \emptyset) = 1$$

Corollary The survival probability of 1s is a left-continuous function in r on $[0, r_c)$ and continuous in r on $[r_c, \infty)$.

Once we know these two last results, we can state

Theorem **Existence and Uniqueness of a critical value for r.**
There exists a unique $r_c \in (0, \infty)$ such that if $r < r_c$, then the 1s have a strictly positive probability to survive, while if $r \geq r_c$, then the 1s die out almost surely.

5. Mean-field model

It is a deterministic non-spatial corresponding model for densities u_i, $i \in F$ for species. The transition rates of the process become a system of differential equations,

$$\begin{cases} u_1' = (\lambda_1 u_1 + \lambda_2 u_3)u_0 + u_3 - u_1(r+1) \\ u_2' = r u_0 + u_3 - u_2 \\ u_3' = r u_1 - 2u_3 \end{cases}$$

with $u_0 + u_1 + u_2 + u_3 = 1$. There are two trivial equilibria : $(0, 0, 0)$, and $(0, r/(r+1), 0)$ which is unstable iff

$$2\lambda_1 + \lambda_2 r > (r+2)(r+1)$$

Perspectives

⟶ Hydrodynamic limit of the process.

⟶ Behaviour of the process on the homogeneous tree T^d, first for $\lambda_2 = 0$ (contact process in dynamic environment), then for $\lambda_2 > 0$.

References

[1] R. Durrett. *Lecture Notes on Particle Systems and Percolation*. Wadsworth, Pacific Grove, CA, 1988.
[2] R. Durrett. *École D'Été De Probabilités De Saint-Flour XXIII*. Lecture Notes in Mathematics 1608, Springer-Verlag, 1993.
[3] T.E. Harris. Contact interactions on a lattice. *Ann. Probab.*, 2 :969–988, 1974.
[4] T. M. Liggett. *Interacting Particle Systems*. Springer-Verlag, 1985.
[5] T.M. Liggett. *Stochastic interacting systems : contact, voter, and exclusion processes*. Springer-Verlag, 1999.
[6] A.M. Van der Vloedt and W. Klassen. The development and application of the sterile insect technique (sit) for new world screwworm eradication. *World Animal Review*, pages 42–49, 1991.