Some generalizations of the KPZ equation

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Outline

KPZ equation

- KPZ equation, after Kardar-Parisi-Zhang '86:

$$\partial_t h = D(\rho)\Delta h + \tfrac{a}{2}\lambda(\rho)(\nabla h)^2 + \sigma(\rho)\mathcal{W}_t,$$

  where $\mathcal{W}$ is a space-time white noise
- Fluctuations of a flat, 1-d nonlinear interface
- Solutions are locally Brownian $\longrightarrow$ ill-posed problem
- ρ: slope of the interface
- Fluctuation-dissipation relation: $\sigma^2(\rho) = 2\chi(\rho)D(\rho)$
- Einstein relation: $\lambda(\rho) = \frac{d}{d\rho}(\chi(\rho)D(\rho))$ [Gonçalves,J.'10]

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Stochastic Burgers equation

- Formally, $\mathcal{Y} = -\nabla h$ leads to the stochastic Burgers equation

$$d\mathcal{Y}_t = D\Delta\mathcal{Y}_t dt + \tfrac{a}{2}\lambda\nabla\mathcal{Y}_t^2 dt + \sigma\nabla\mathcal{W}_t$$

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- Fluctuations of a conserved quantity on 1-d nonlinear systems, near a stationary state

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Stochastic heat equation

- Itô formula for $z(t,x) = e^{\gamma h(t,x)}$ not true!!!, $\gamma = \frac{2D}{a\lambda}$ gives

$$\partial_t z = D\Delta z + \frac{a\lambda\sigma}{D} z\mathcal{W}$$

- Stochastic heat equation $\longrightarrow$ well-posed!!!
- $h = \frac{1}{\gamma}\log z$, Cole-Hopf solution of KPZ equation
- "Physical" solutions, according to [KPZ, PRL '86]
- Justified by [Bertini-Giacomin, CMP '97] $\longrightarrow$ derivation of Cole-Hopf solutions from a microscopic dynamics $\longrightarrow$ very model-dependent
- WASEP $\longrightarrow$ go to the white-board!!!

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A little of recent history

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- Amir-Corwin-Quastel arXiv:1003.0443
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Second-order Boltzmann-Gibbs principle

- Main technical innovation: two-blocks estimate at the level of fluctuations, without averaging
- Spectral gap + Kipnis-Varadhan + equivalence of ensembles + renormalization
- Robust scheme, not model-dependent [Gonçalves, J.,Sethuraman '12]
- Notion of energy solutions of KPZ-SBE equation

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Conjecture

Stationary energy solutions of the stochastic Burgers equation are unique in distribution

Theorem (Gonçalves, J. '10, Gonçalves, J.,Sethuraman '12)

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Universality of KPZ equation

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Fractional KPZ equation

Consider the fractional stochastic Burgers equation

$$d\mathcal{Y}_t = -(-\Delta)^{\theta}\mathcal{Y}_t dt + \nabla \mathcal{Y}_t^2 dt + (-\Delta)^{\theta/2} d\mathcal{W}_t \qquad (1)$$

Theorem (Gubinelli, J. '12)

- *If $\theta > 1$, there exist stationary energy solutions of* (1)
- *If $\theta > 5/4$ there is at most one stationary energy solution of* (1)

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- Da Prato-Debussche-Tubaro '07 $\rightarrow$ uniqueness for $\theta > \frac{1}{8}$
- Existence for $\theta > -\frac{1}{8}$
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- Universality result for Sasamoto-Spohn model

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Exclusion process with long jumps

- Why fractional SBE???
- Exclusion process with long jumps, introduced in [J., CPAM '08] $\rightarrow$ see the white-board
- Along subsequences, density fluctuations converge to energy solutions of the fractional SBE
- $\theta = 3/4$ is special: if uniqueness holds, we have a 1:2:3 scale-invariant process $\rightarrow$ new universality class???
- Conjecture: anomalous heat equation belongs to fKPZ universality class

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Systems of KPZ equations

- What if two or more conserved quantities???
- Two or more KPZ equations!!!

$$d\mathcal{Y}_t^i = D_i \Delta \mathcal{Y}_t^i dt + \nabla \mathcal{Q}^i(\mathcal{Y}) dt + \nabla \mathcal{W}_t^i$$

$\mathcal{Y}_t = (\mathcal{Y}_t^1, \dots, \mathcal{Y}_t^\ell)$, $\mathcal{Q}^i$, quadratic forms on $\mathcal{Y}$
$\mathcal{W}_t = (\mathcal{W}_t^1, \dots, \mathcal{W}_t^\ell)$ a (possibly correlated) white noise

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- Concrete example: Bernardin-Stolz model
- Not-So-Strongly-Stochastically perturbed Hamiltonian chain $\longrightarrow$ see the white-board
- Harmonic potential ($V(r) = \frac{1}{2}r^2$):

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