Additive functionals of exclusion processes

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Interacting Particle Systems and Related Topics
Florence, Italy
August 2012
(joint work with Milton Jara - IMPA)

1. Abstract

I consider exclusion processes evolving on $\mathbb{Z}$ and starting from the invariant state. The goal consists in establishing scaling limits of $\Gamma_t(f) := \int_0^t f(\eta_s)ds$ for local functions f. I present a method, recently introduced in [3], from which we derive a local Boltzmann-Gibbs Principle for a class of exclusion processes. For the occupation time of the origin (i.e. for $f(\eta) := \eta(0)$), this principle says that $\Gamma_t(f)$ is very well approximated to the density of particles. As a consequence, the scaling limits of $\Gamma_t(f)$ follow from the scaling limits of the density of particles. As examples I present the mean-zero exclusion, the symmetric simple exclusion and the weakly asymmetric simple exclusion. For the latter under a strong asymmetry, we establish the limit of $\Gamma_t(f)$ in terms of the solution of the KPZ equation.

2. Lattice gas dynamics

- η_t is a Markov process with space state $\Omega := \{0,1\}^{\mathbb{Z}}$.
- for $x \in \mathbb{Z}$, $\eta(x) = 1$ if the site x is occupied, otherwise $\eta(x) = 0$.
- Let $r : \Omega \to \mathbb{R}$ be a local function that satisfies:

i) There exists $\varepsilon_0 > 0$ such that $\varepsilon_0 < r(\eta) < \varepsilon_0^{-1}$ for any $\eta \in \{0,1\}^{\mathbb{Z}}$. (**Ellipticity**)
ii) For any η, ξ such that $\eta(x) = \xi(x)$ for $x \neq 0, 1$, then $r(\eta) = r(\xi)$. (**Reversibility**)

- The generator is given on local functions $f : \Omega \to \mathbb{R}$ by:

$$\mathcal{L}f(\eta) = \sum_{x \in \mathbb{Z}} r(\tau_x \eta)(f(\eta^{x,x+1}) - f(\eta))$$

where

$$\eta^{x,y}(z) = \begin{cases} \eta(y), & z = x \\ \eta(x), & z = y \\ \eta(z), & z \neq x, y. \end{cases} \tag{1}$$

Invariant states

Let ν_ρ be the Bernoulli product measures of constant parameter $\rho \in [0,1]$. Under this measure the occupation variables $(\eta(x))_x$ are independent and

$$\nu_\rho(\eta : \eta(x) = 1) = \rho.$$

We denote by $\mathbb{E}_\rho$ the expectation with respect to $\mathbb{P}_\rho$ - the distribution of $\{\eta_t : t \geq 0\}$ in the space $\mathcal{D}([0,\infty], \Omega)$ starting from ν_ρ.

3. Scaling Limits of the density of particles

- **Hydrodynamic Limit**

For each configuration η we denote by $\pi^n(\eta; du)$ the empirical measure:

$$\pi^n(\eta; du) = \frac{1}{n} \sum_{x \in \mathbb{Z}} \eta(x) \delta_{x/n}$$

and $\pi^n_t(\eta, du) := \pi^n(\eta_t, du)$.
Under a diffusive scaling of time tn^2, the hydrodynamic limit (LLN) was obtained by [2].

- **Equilibrium fluctuations**:

The density fluctuation field acts on functions $G \in \mathcal{S}(\mathbb{R})$ as

$$\mathcal{Y}^n_t(G) := \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} G\Big(\frac{x}{n}\Big)\{\eta_{tn^2}(x) - \rho\}. \tag{2}$$

It was proved by [1] that $\{\mathcal{Y}^n_t; t \in [0,T]\}$ converges in distribution with respect to the Skorohod topology of $\mathcal{D}([0,T], \mathcal{S}'(\mathbb{R}))$ to the stationary solution of the Ornstein-Uhlenbeck equation

$$d\mathcal{Y}_t = D(\rho)\Delta \mathcal{Y}_t dt + \sqrt{2D(\rho)\rho(1-\rho)}\nabla d\mathcal{B}_t, \tag{3}$$

where $\mathcal{B}_t$ is a $\mathcal{S}'(\mathbb{R})$-valued Brownian motion and $D(\rho)$ is the diffusion coefficient.

This means that the trajectories of the limit field $\mathcal{Y}_t$ are in $\mathcal{C}([0,T], \mathcal{S}'(\mathbb{R}))$ and that $\mathcal{Y}_0$ is a white noise of variance $\rho(1-\rho)$ - namely if for any $G \in \mathcal{S}(\mathbb{R})$, the real-valued random variable $\mathcal{Y}_0(G)$ has a normal distribution of mean zero and variance $\rho(1-\rho)\|G\|^2$.

Now, fix a stationary solution $\{\mathcal{Y}_t; t \in [0,T]\}$ of (3).
For $x \in \mathbb{R}$, let $i_\varepsilon(x) : y \mapsto \varepsilon^{-1} 1_{(0,1]}((y-x)\varepsilon^{-1})$.

Theorem 1: *For each $\varepsilon \in (0,1)$, let $\{\mathcal{Z}^\varepsilon_t; t \in [0,T]\}$ be defined as*

$$\mathcal{Z}^\varepsilon_t = \int_0^t \mathcal{Y}_s(i_\varepsilon) ds.$$

Then, the process $\{\mathcal{Z}^\varepsilon_t; t \in [0,T]\}$ converges in distribution with respect to the uniform topology of $\mathcal{C}([0,T], \mathbb{R})$, as $\varepsilon \to 0$, to a fractional Brownian motion $\{\mathcal{Z}_t; t \in [0,T]\}$ of Hurst exponent $H = 3/4$.

4. Additive functionals

Our goal consists in obtaining functional limit theorems for observables of the process $\{\eta_t; t \geq 0\}$. The occupation time of a site $x \in \mathbb{Z}$ is defined as the integral $\int_0^t \eta_s(x) ds$.
More generally, for $f : \Omega \to \mathbb{R}$ a local function, if for $\beta \in [0,1]$, $\varphi_f(\beta) = \int f d\nu_\beta$, then:

Theorem 2: *The process $\{\Gamma^n_t(f); t \in [0,T]\}$ defined as*

$$\Gamma^n_t(f) = \frac{1}{n^{3/2}} \int_0^{tn^2} \big(f(\eta_s) - \varphi_f(\rho)\big) ds \tag{5}$$

converges in distribution with respect to the uniform topology of $\mathcal{C}([0,T], \mathbb{R})$ to $\{\varphi'_f(\rho)\mathcal{Z}_t; t \in [0,T]\}$, where $\{\mathcal{Z}_t; t \in [0,T]\}$ is the same as above.

5. Strategy of the proof

Theorem 3: *Local Boltzmann-Gibbs principle*. *Let $f : \Omega \to \mathbb{R}$ be a local function, such that $supp(f) \subseteq \{1, ..., k\}$ and $\varphi_f(\rho) = 0$. There exists $c = c(f, \rho)$ such that*

i) if $\varphi'_f(\rho) \neq 0$, then for any $t \geq 0$ and any $\ell \geq k$:

$$\mathbb{E}_\rho\Big[\Big(\int_0^t \Big\{f(\eta_s) - \varphi'_f(\rho)\big(\eta^\ell_s - \rho\big)\Big\} ds\Big)^2\Big] \leq c\Big(t\ell + \frac{t^2}{\ell^2}\Big),$$

ii) if $\varphi'_f(\rho) = 0$, then for any $t \geq 0$ and any $\ell \geq k$:

$$\mathbb{E}_\rho\Big[\Big(\int_0^t \Big\{f(\eta_s) - \frac{\varphi''_f(\rho)}{2}\Big(\big(\eta^\ell_s - \rho\big)^2 - \frac{\rho(1-\rho)}{\ell}\Big)\Big\} ds\Big)^2\Big] \leq c\Big(t(\log \ell)^2 + \frac{t^2}{\ell^3}\Big)$$

where $\eta^\ell := \frac{1}{\ell}\sum_{x=1}^{\ell} \eta(x)$.

The proof of last result is divided into four steps that rely on the Kipnis-Varadhan inequality (see [5]) and the spectral gap inequality (see [6]):

1. Firstly, we compare the additive functional associated to f with the additive functional associated to $\psi_f(\ell) := E_\rho[f|\sum_{x=1}^{\ell} \eta(x)]$, using the:

Lemma 1: *One-block estimate*. *Let $f : \Omega \to \mathbb{R}$ be a local function such that $\varphi_f(\rho) = 0$. Then, there exists $c = c(f, \rho)$ such that for any $\ell \geq k$ and any $t \geq 0$:*

$$\mathbb{E}_\rho\Big[\Big(\int_0^t \{f(\eta_s) - \psi_f(\ell; \eta_s)\} ds\Big)^2\Big] \leq ct\ell^2 \,\boldsymbol{Var}(f; \nu_\rho).$$

2. Secondly, we compare the additive functional associated to $\psi_f(\ell)$ with the additive functional associated to $\psi_f(2\ell)$, using the:

Lemma 2: *Renormalization step*. *Let $f : \Omega \to \mathbb{R}$ be a local function such that $\varphi_f(\rho) = 0$. There exists $c = c(f, \rho)$ such that for any $\ell \geq k$ and any $t \geq 0$:*

$$\mathbb{E}_\rho\Big[\Big(\int_0^t \{\psi_f(\ell; \eta_s) - \psi_f(2\ell; \eta_s)\} ds\Big)^2\Big] \leq \begin{cases} ct\ell, \text{ if } \varphi'_f(\rho) \neq 0, \\ ct, \text{ if } \varphi'_f(\rho) = 0. \end{cases}$$

3. Thirdly, we compare the additive functional associated to $\psi_f(k)$ with the additive functional associated to $\psi_f(2^m k)$, using the renormalization step m times.

Lemma 3: *Two-blocks estimate*. *Let $f : \Omega \to \mathbb{R}$ be a local function such that $\varphi_f(\rho) = 0$. Then, there exists $c = c(f, \rho)$ such that for any $\ell \geq k$ and any $t \geq 0$:*

$$\mathbb{E}_\rho\Big[\Big(\int_0^t \psi_f(k; \eta_s) - \psi_f(\ell; \eta_s) ds\Big)^2\Big] \leq \begin{cases} ct\ell, \text{ if } \varphi'_f(\rho) \neq 0, \\ ct(\log \ell)^2, \text{ if } \varphi'_f(\rho) = 0. \end{cases}$$

4. Finally, we replace $\psi_f(\ell)$ by the corresponding function of η^ℓ using the following:

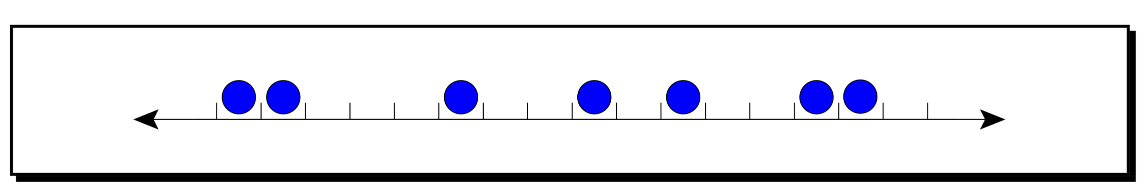
Proposition 1: *Equivalence of Ensembles*. *Let $f : \Omega \to \mathbb{R}$ be a local function. Then there exists a constant $c = c(f, \rho)$ such that for any $\ell \geq k$:*

$$\int \Big(\psi_f(\ell, \eta) - \varphi'_f(\rho)\big(\eta^\ell - \rho\big) - \frac{\varphi''_f(\rho)}{2}\Big(\big(\eta^\ell_s - \rho\big)^2 - \frac{\rho(1-\rho)}{\ell}\Big)\Big)^2 d\nu_\rho \leq \frac{c}{\ell^3}.$$

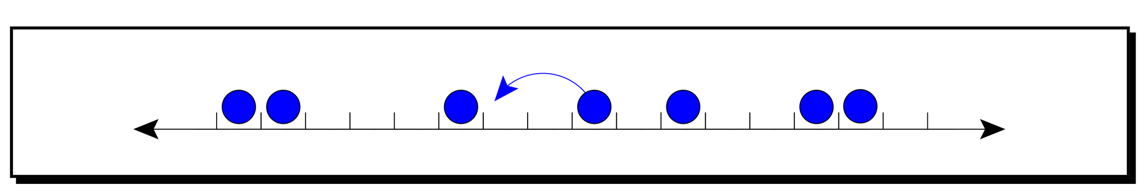
6. Examples

1-Mean-Zero Exclusion

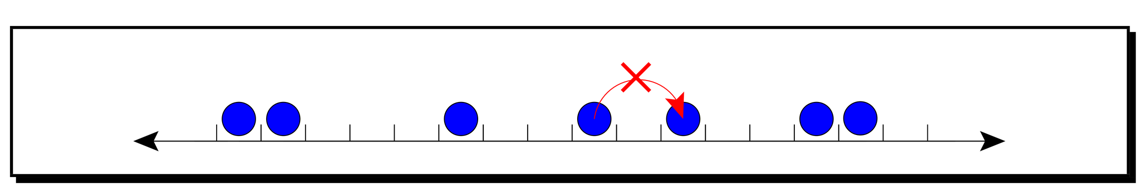
Initially one can have a certain number of particles, as for example:



After an exponential time of parameter 1, a particle at x jumps to $x + y$ with probability $p(y)$, but to respect the exclusion rule, particles can only jump to empty sites. So, this jump is allowed:



But, this jump is forbidden:



The microscopic dynamics:

Let $p : \mathbb{Z} \setminus \{0\} \to [0,1]$ be a probability measure, such that:

1. $p(\cdot)$ has finite range, that is, there exists $M > 0$ such that $p(z) = 0$ whenever $|z| > M$;
2. $p(\cdot)$ is irreducible, i.e. $\mathbb{Z} = \text{span}\{z \in \mathbb{Z}; p(z) > 0\}$;
3. $p(\cdot)$ has *mean zero*: $\sum_{z \in \mathbb{Z}} zp(z) = 0$.

Example: $p(1) = 2/3$, $p(-2) = 1/3$ **and** $p(z) = 0$ **if** $z \neq -2, 1$.

We define the Markov process $\{\eta^{\text{ex}}_t; t \geq 0\}$ whose **generator** acts over local functions $f : \Omega \to \mathbb{R}$ as

$$\mathcal{L}_{\text{ex}} f(\eta) = \sum_{x,y \in \mathbb{Z}} p(y)\eta(x)(1 - \eta(x+y))(f(\eta^{x,x+y}) - f(\eta)),$$

with $p(\cdot)$ as above and $\eta^{x,x+y}$ as in (1).

The measures $\{\nu_\rho; \rho \in [0,1]\}$ are **invariant**, but they are **not necessarily reversible** (that is true if and only if $p(\cdot)$ is symmetric). The mean-zero, non-symmetric exclusion process is a **diffusive** and **non-reversible system**.

We can define the density fluctuation field $\{\mathcal{Y}^n_t; t \in [0,T]\}$ as in (2) and we have that:

Proposition 2: *The process $\{\mathcal{Y}^n_t; t \in [0,T]\}$ converges in distribution with respect to the Skorohod topology of $\mathcal{D}([0,T], \mathcal{S}'(\mathbb{R}))$ to the stationary solution of the Ornstein-Uhlenbeck equation*

$$d\mathcal{Y}_t = D(\rho)\Delta \mathcal{Y}_t dt + \sqrt{2D(\rho)\rho(1-\rho)(\rho)}\nabla d\mathcal{B}_t,$$

where $D(\rho)$ is the diffusion coefficient.

The results presented above allow us to get the scaling limits of additive functionals as in Theorem 2.

2-Symmetric simple exclusion

Let p be such that $p(1) = p(-1) = 1/2$. The measures $\{\nu_\rho; \rho \in [0,1]\}$ are **invariant**.

Proposition 3: *The process $\{\mathcal{Y}^n_t; t \in [0,T]\}$ converges in distribution with respect to the Skorohod topology of $\mathcal{D}([0,T], \mathcal{S}'(\mathbb{R}))$ to the stationary solution of the Ornstein-Uhlenbeck equation*

$$d\mathcal{Y}_t = \frac{1}{2}\Delta \mathcal{Y}_t dt + \sqrt{\rho(1-\rho)}\nabla d\mathcal{B}_t.$$

The results presented above allow us to get the scaling limits of additive functionals as in Theorem 2.

3-The weakly asymmetric simple exclusion

Let $p_n(1) = \frac{1}{2} + \frac{a_n}{2}$, $p_n(-1) = \frac{1}{2} - \frac{a_n}{2}$ and $p_n(z) = 0$ if $z \neq -1, 1$. The measures $\{\nu_\rho; \rho \in [0,1]\}$ are **invariant**. If $a_n := \frac{1}{n}$, then we have that

Proposition 4: *The process $\{\mathcal{Y}^n_t; t \in [0,T]\}$ converges in distribution with respect to the Skorohod topology of $\mathcal{D}([0,T], \mathcal{S}'(\mathbb{R}))$ to the stationary solution of the Ornstein-Uhlenbeck equation*

$$d\mathcal{Y}_t = \frac{1}{2}\Delta \mathcal{Y}_t dt + (1-2\rho)\nabla \mathcal{Y}_t dt + \sqrt{\rho(1-\rho)}\nabla d\mathcal{B}_t.$$

In this case the Ornstein-Uhlenbeck process has a drift, nevertheless one can get the same result as in Theorem 2.

4-The weakly asymmetric simple exclusion, the KPZ scaling

If $\rho = 1/2$, then the limit field is the same as in the symmetric simple exclusion (so a weak asymmetry does not have influence!), see [4]. In this case the "correct" **strength asymmetry** is $a_n = 1/\sqrt{n}$. In this case we have

Proposition 5: *The process $\{\mathcal{Y}^n_t; t \in [0,T]\}$ converges in distribution with respect to the Skorohod topology of $\mathcal{D}([0,T]; \mathcal{S}'(\mathbb{R}))$ to the stationary solution of the stochastic Burgers equation*

$$d\mathcal{Y}_t = \frac{1}{2}\Delta \mathcal{Y}_t dt + \big(\nabla \mathcal{Y}_t\big)^2 dt + \sqrt{\rho(1-\rho)}\nabla d\mathcal{B}_t. \tag{7}$$

Theorem 4: *Let $\{\mathcal{Y}_t; t \in [0,T]\}$ the stationary solution of the stochastic Burgers equation above. For $\varepsilon > 0$, let $\mathcal{Z}^\varepsilon_t = \int_0^t \mathcal{Y}_s(i_\varepsilon)ds$. Then there exists $\{\mathcal{Z}_t; t \in [0,T]\}$ such that $\{\mathcal{Z}^\varepsilon_t; t \in [0,T]\}$ converges in distribution with respect to the uniform topology of $\mathcal{C}([0,T]; \mathbb{R})$, as $\varepsilon \to 0$, to $\{\mathcal{Z}_t; t \in [0,T]\}$.*

And as a consequence we have that

Theorem 5: *Let $f : \Omega \to \mathbb{R}$ be a local function such that $\varphi_f(1/2) = 0$. Then, $\{\Gamma^n_t(f); t \in [0,T]\}$ as defined in (5) converges in distribution with respect to the uniform topology of $\mathcal{C}([0,T]; \mathbb{R})$ to $\{\varphi'_f(1/2)\mathcal{Z}_t; t \in [0,T]\}$, where $\mathcal{Z}_t$ is the same as in Theorem 4.*

5-Symmetric simple exclusion/Asymmetric simple exclusion

Let f be a local function. Then

- if $\varphi_f(\rho) = 0$, $\varphi'_f(\rho) \neq 0$, then $\mathbf{Var}(\Gamma_t(f); \nu_\rho) \leq t^{3/2}(t^{4/3}, \rho = 1/2; t, \rho \neq 1/2)$,

  With the results presented above we can get the correct upper bound in the symmetric case. Our method does not give the correct upper bound in the asymmetric case.

- if $\varphi_f(\rho) = \varphi'_f(\rho) = 0$, $\varphi''_f(\rho) \neq 0$, then $\mathbf{Var}(\Gamma_t(f); \nu_\rho) \leq t\log(t)(t)$,

  With the results presented above we can get the upper bound $t(log(t))^2$ in the symmetric case. Our method does not give the correct upper bound in the asymmetric case.

- if $\varphi_f(\rho) = \varphi'_f(\rho) = \varphi''_f(\rho) = 0$, $\varphi'''_f(\rho) \neq 0$, then $\mathbf{Var}(\Gamma_t(f); \nu_\rho) \leq t$.

  With the results presented above we can get the correct upper bound in both cases, see also [7].

7. References

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