Anomalous current fluctuations at a phase transition

IPS Workshop / Villa Finaly

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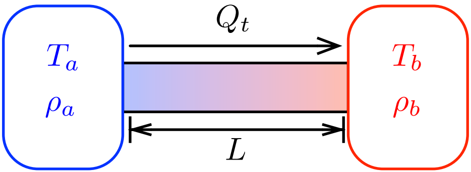
Fourier's law



Q_t : integrated current of energy/particles during time t.

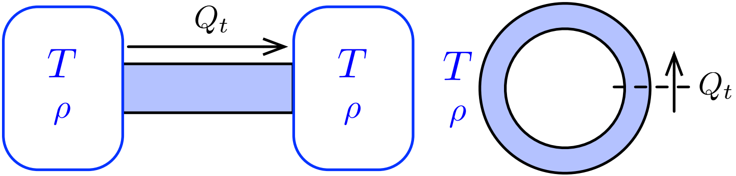
Fourier's law (energy) / Fick's law (particles) :

$$\frac{\langle Q_t \rangle}{t} \propto \frac{1}{L}$$

Statistics of Q_t

- cumulants $\langle Q_t^n \rangle_c$
- generating function $\log \langle e^{\lambda Q_t} \rangle$
- large deviation function $\mathrm{Pro}[Q_t \simeq qt] \sim e^{-t\mathcal{F}(q)}$

Non-trivial even at equilibrium :



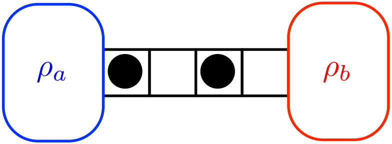
From fluctuation-dissipation, we expect

$$\frac{\langle Q_t^2 \rangle_c}{t} \propto \frac{1}{L}$$

How do the $\langle Q_t^n \rangle_c$ behave ?

Diffusive systems

Diffusive systems (symmetric exclusion processes, etc.)
obey Fourier's law



Open geometry :

$$\frac{1}{t}\log\left\langle e^{\lambda Q_t}\right\rangle \simeq \frac{1}{L}\mathcal{F}(\rho_a,\rho_b,\lambda)$$

All cumulants decay in $1/L$:

$$\frac{\langle Q_t^n\rangle}{t} \propto \frac{1}{L}$$

Bodineau, Derrida (2004)

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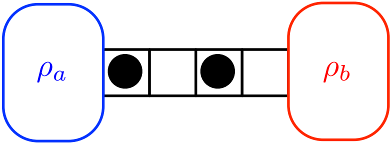

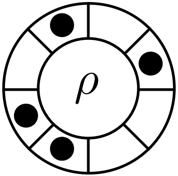

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Bodineau, Derrida (2004)

Ring geometry :

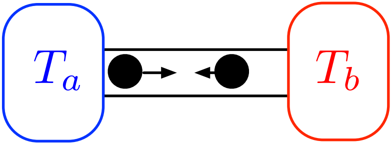
$$\frac{\log\left\langle e^{\lambda Q_t}\right\rangle}{t} \simeq \frac{\alpha\lambda^2}{L} + \frac{1}{L^2}\mathcal{F}(\rho,\lambda)$$

Higher cumulants in $1/L^2$

Appert, Derrida, Lecomte, van Wijland (2008)

Mechanical models

1D momentum-conserving models have an anomalous Fourier's law



Hard particle gas :

$$\frac{\langle Q_t \rangle}{t} \propto \frac{1}{L^{2/3}}$$

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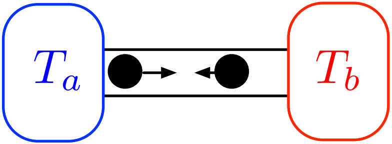

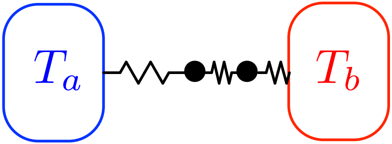

Hard particle gas :

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Anharmonic chain (Fermi-Pasta-Ulam 'β') :

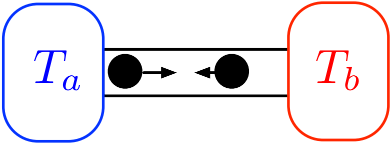
$$\frac{\langle Q_t \rangle}{t} \propto \frac{1}{L^{\sim 3/5}}$$

$\Rightarrow$ Anomalous conduction with two different regimes
Delfini, Lepri, Livi, Politi, van Beijeren, Spohn...

Mechanical models

For the HPG, this behavior extends to the cumulants :

Brunet Derrida G. (2010)



Open geometry : similar decay

$$\frac{\langle Q_t^n \rangle}{t} \propto \frac{1}{L^{2/3}}$$

for $n \leq 4$

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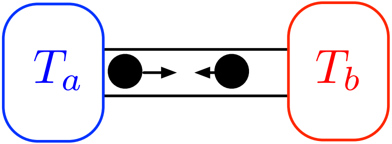

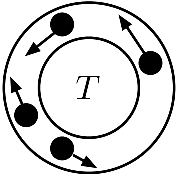

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$$\frac{\langle Q_t^n \rangle}{t} \propto \frac{1}{L^{2/3}}$$

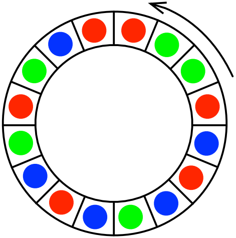
for $n \leq 4$

Ring geometry :

$$\frac{\langle Q_t^2 \rangle_c}{t} \propto \frac{1}{\sqrt{L}} \,, \quad \frac{\langle Q_t^4 \rangle_c}{t} \propto \sqrt{L}$$

Higher cumulants seem to grow faster with L

In this talk : the ABC model

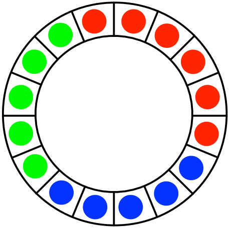


$$AB \underset{1}{\overset{q}{\rightleftharpoons}} BA$$

$$BC \underset{1}{\overset{q}{\rightleftharpoons}} CB$$

$$CA \underset{1}{\overset{q}{\rightleftharpoons}} AC$$

$$q < 1$$



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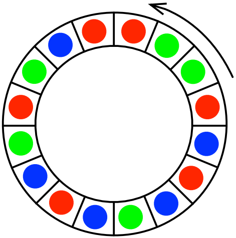

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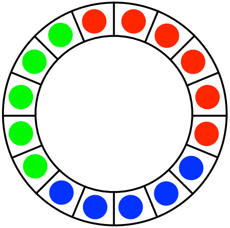
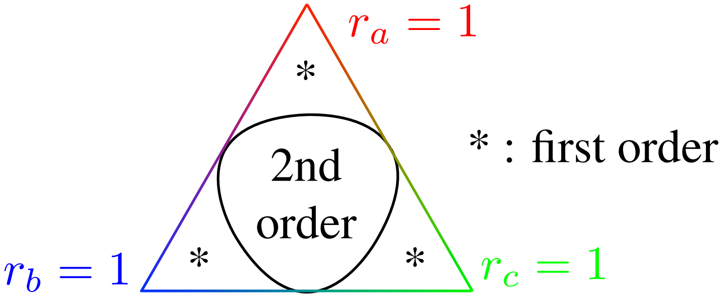


- When $q = e^{-\beta/L}$: phase transition at $\beta = \beta_*(r_a, r_b, r_c)$
- Anomalous current fluctuations around β_* :

$$\frac{\langle Q_a(t)\rangle}{t} = \frac{J}{L} + \frac{\text{corr.}}{L^{3/2}} \;,\; \frac{\langle Q_a^n(t)\rangle_c}{t} \propto L^{n-5/2}$$

Phase diagram (on a ring)



- Transition can be first or second-order, depending on r_a, r_b, r_c
- If it is second-order, then it occurs at

$$\beta_* = \frac{2\pi}{\sqrt{1 - 2\sum r_i^2}} \, .$$

- β_* is unknown otherwise

The second-order transition

- The ABC model obeys diffusive dynamics :

$$\mathrm{Pro}[\text{site } k \text{ is of type } a] \simeq \rho_a\left(k/L, t/L^2\right) = \rho_a(x,\tau)$$

$$\begin{cases} \partial_\tau \rho_a = -\partial_x j_a & \text{(conservation)} \\ j_a = -\partial_x \rho_a + \beta\rho_a(\rho_c - \rho_b) & \text{(biased Fick's law)} \end{cases}$$

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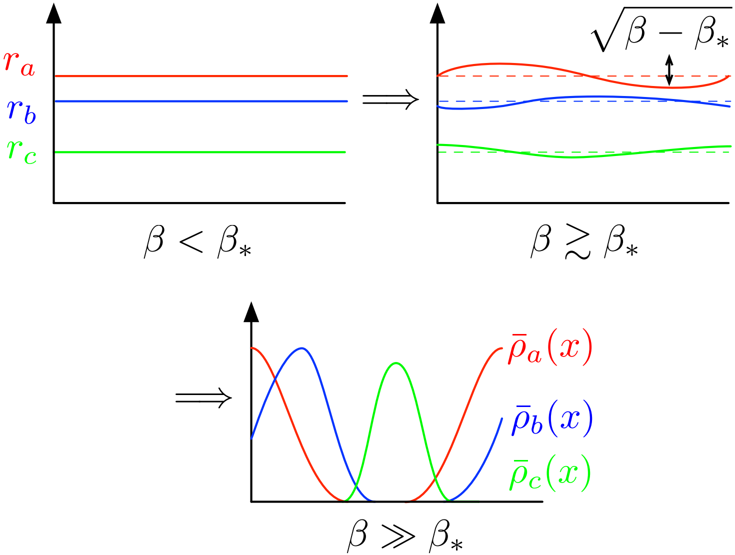
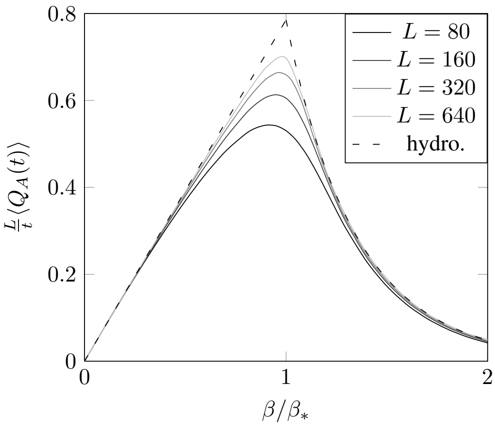
- the constant profiles $\rho_a(x,\tau) = r_a$ are stable for $\beta < \beta_*$
  $\Rightarrow$ homogeneous / disordered phase

- They become unstable for $\beta \geq \beta_*$: the new stable profiles are modulated in space
  $\Rightarrow$ modulated / ordered phase

The second-order transition

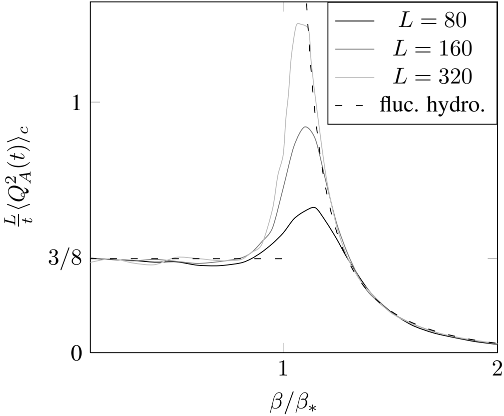


First cumulant of the current



$$r_A = r_C = 1/4$$
$$r_B = 1/2$$

Second cumulant



$r_A = r_C = 1/4$

$r_B = 1/2$

Critical regime

- The first Fourier mode of ρ_a becomes unstable as $\beta \uparrow \beta_*$
- Assume that, for $\beta \simeq \beta_*$, this mode, R_a, varies slowly and is larger than the others :

$$\rho_a(x,\tau) = r_a + (R_a(\tau)e^{2i\pi x} + cc.) + o(R_a)$$

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$$\rho_a(x,\tau) = r_a + (R_a(\tau)e^{2i\pi x} + cc.) + o(R_a)$$

- The hydrodynamics equations $j_a = -\partial_x \rho_a + \beta \rho_a(\rho_c - \rho_b)$ give the effective dynamics of R_a :

$$\frac{dR_a}{d\tau} = 4\pi^2 \left[\gamma - \frac{2\Lambda}{\Delta^2}|R_a|^2\right] R_a$$

with

$$\gamma = \frac{\beta - \beta_*}{\beta_*} \qquad \Delta = 1 - 2\sum_a r_a^2 \qquad \Lambda = \sum_a r_a^2 - 2\sum_a r_a^3$$

Fluctuating hydrodynamics

- Fluctuations of Q_t arise from stochastic corrections to the hydrodynamics :

$$j_a = -\partial_x \rho_a + \beta \rho_a(\rho_c - \rho_b) + \frac{\eta_a(x,\tau)}{\sqrt{L}}$$

- $\eta_a(x,t)$: white noise $\langle \eta_a(x,\tau)\eta_b(x',\tau')\rangle = \sigma_{ab}\delta(x-x')\delta(\tau-\tau')$

$$\text{with } \sigma_{ab} = \begin{cases} 2\rho_a(1-\rho_a) & \text{if } a=b \\ -2\rho_a\rho_b & \text{otherwise.} \end{cases}$$

Fluctuating critical regime

When adding the stochastic corrections,

$$j_a = -\partial_x \rho_a + \beta \rho_a(\rho_c - \rho_b) + \frac{\eta_a(x,\tau)}{\sqrt{L}}$$

the dynamics of R_a get an added complex white noise

$$\frac{dR_a}{d\tau} = 4\pi^2 \left[\gamma - \frac{2\Lambda}{\Delta^2}|R_a|^2\right] R_a + \frac{\nu_a(\tau)}{\sqrt{L}}$$

with

$$\langle \nu_a(\tau)\nu_a^*(\tau')\rangle = \frac{24\pi^2 r_a^2 r_b r_c}{\Delta}\delta(\tau - \tau')$$

Fluctuating critical regime

- Rescaling :

$$R_a = \frac{1}{L^{1/4}} f(\bar{\tau}) \text{ with } \bar{\tau} = \frac{\tau}{\sqrt{L}}$$

$\Rightarrow$ $f(\bar{\tau})$ diffuses in a quartic potential :

$$\frac{df}{d\bar{\tau}} = 4\pi^2 \left(\bar{\gamma} - \frac{2\Lambda}{\Delta^2} |f|^2 \right) f + \nu(\bar{\tau}) \qquad \text{with } \bar{\gamma} = \sqrt{L} \frac{\beta - \beta_*}{\beta_*}$$

$\Rightarrow$ Critical regime for $|\beta - \beta_*| \sim 1/\sqrt{L}$:

- the first Fourier mode fluctuates in $1/L^{1/4}$ (the others in $1/\sqrt{L}$)
- these fluctuations are on a slow time scale $\bar{\tau} \propto t/L^{5/2}$

Current in the critical regime

$$Q_a(t) = L\int_0^{t/L^2} d\tau \left[\beta r_a(r_c - r_b) + \frac{2\beta}{\sqrt{L}}\frac{r_b - r_c}{r_a}|f(\bar\tau)|^2 + (noise)\right]$$

$$\frac{\langle Q_a(t)\rangle}{t} \simeq \frac{\beta}{L} r_a(r_c - r_b) + \frac{2\beta}{L^{3/2}}\frac{r_b - r_c}{r_a} C_1(\bar\gamma)$$

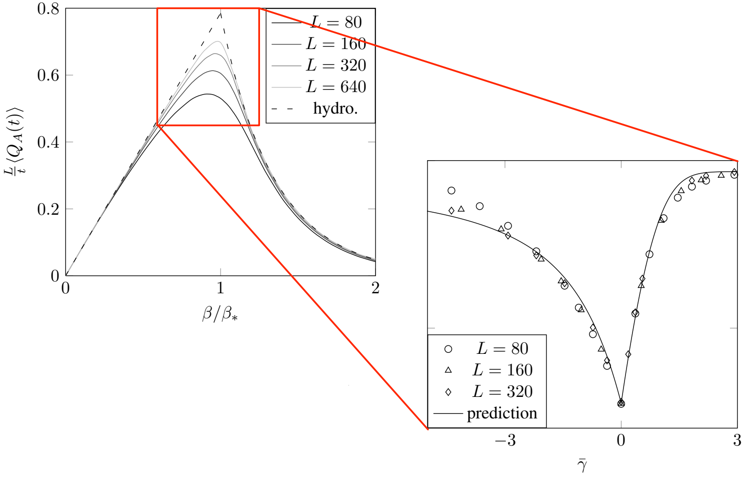
$$\frac{\langle Q_a^n(t)\rangle_c}{t} \simeq \frac{1}{L^{5/2-n}}\left[\frac{2\beta(r_b - r_c)}{r_a}\right]^n C_n(\bar\gamma)$$

- $C_n(\bar\gamma)$ is an *n*-point correlations integral of $|f(\bar\tau)|^2$:

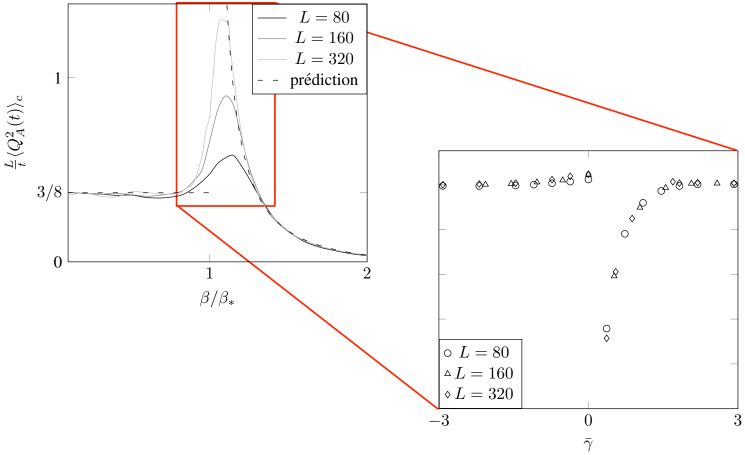
$$C_n(\bar\gamma) = \lim_{\bar\tau\to\infty}\frac{1}{\bar\tau}\int_0^{\bar\tau} d\bar\tau_1..d\bar\tau_n \left\langle |f(\bar\tau_1)..f(\bar\tau_n)|^2\right\rangle$$

(only $C_1 = \left\langle |f|^2\right\rangle$ has an analytical expression)

Rescaling : first cumulant



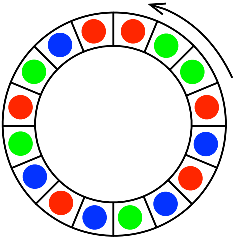
Rescaling : second cumulant



Conclusion

	$\beta \neq \beta_*$	$\beta = \beta_*$	
		$\Lambda \neq 0$	$\Lambda = 0$
Time scale	t/L^2	$t/L^{5/2}$	$t/L^{8/3}$
Cumulants	$\frac{\langle Q_t^n \rangle_c}{t} \propto \frac{1}{L}$	$\frac{\langle Q_t^n \rangle_c}{t} \propto L^{n-5/2}$	$\frac{\langle Q_t^n \rangle_c}{t} \propto L^{4(n-2)/3}$
Correlations	$\langle \rho(x)\rho(y) \rangle_c \propto \frac{1}{L}$	$\langle \rho(x)\rho(y) \rangle_c \propto \frac{1}{\sqrt{L}}$	$\langle \rho(x)\rho(y) \rangle_c \propto \frac{1}{L^{1/3}}$

Why a phase transition ?



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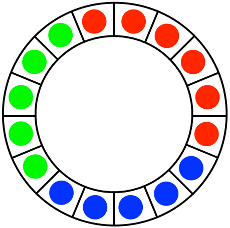

- For $r_a = r_b = r_c = 1/3$, there is detailed balance with energy

$$E = \sum_{i=1}^{L-1} \sum_{j=i+1}^{L} [C_i B_j + A_i C_j + B_i A_j] \qquad A_i = \begin{cases} 1 \text{ if } i \text{ is of type } A \\ 0 \text{ otherwise} \end{cases}$$

Evans (M.) Kafri Koduvely Mukamel (1998)

$\Rightarrow$ effective interactions are long-ranged

- For generic r_a, r_b, r_c, non-equilibrium steady state

Tricritical regime

- First mode evolution equation

$$\frac{dR_a}{d\tau} = 4\pi^2 \left[\gamma - \frac{2\Lambda}{\Delta^2} |R_a|^2 \right] R_a + \frac{\nu_a(\tau)}{\sqrt{L}}$$

- $\Lambda = \sum r_a^2 - 2 \sum r_a^3$ vanishes on the tricritical line :

$$\begin{cases} \Lambda \geq 0 \Rightarrow \text{second-order transition} \\ \Lambda < 0 \Rightarrow \text{first-order transition} \end{cases}$$

- Going to the next order yields

$$\frac{dR_a}{d\tau} = 4\pi^2 \left[\gamma - \frac{|R_a|^4}{r_a^2 \Delta} \right] R_a + \frac{\nu_a(\tau)}{\sqrt{L}}$$

Scalings and current fluctuations

- Different scaling for R_a :

$$R_a = \frac{1}{L^{1/6}} g(\tilde{\tau}) \text{ with } \tilde{\tau} = \frac{\tau}{L^{2/3}}$$

$\Rightarrow$ Larger fluctuations on a slower time scale

- Faster growth of the cumulants :

$$\frac{\langle Q_a(t) \rangle}{t} \simeq \frac{\beta}{L} r_a (r_c - r_b) + \frac{2\beta}{L^{4/3}} \frac{r_b - r_c}{r_a} D_1(\bar{\gamma})$$

$$\frac{\langle Q_a^n(t) \rangle_c}{t} \simeq L^{4(n-2)/3} \left[\frac{2\beta(r_b - r_c)}{r_a} \right]^n D_n(\bar{\gamma})$$