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Anomalous current fluctuations at a phase transition IPS Workshop / Villa Finaly

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Fourier's law

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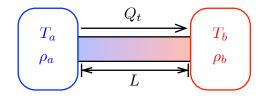
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phase transition

Fourier's law

- Diffusive systems Mechanical mode ABC model Phase transitio
- Critical regime
- Conclusion



 Q_t : integrated current of energy/particles during time t. Fourier's law (energy) / Fick's law (particles) :

$$rac{\langle Q_t
angle}{t} \propto rac{1}{L}$$

Introduction

Fourier's law

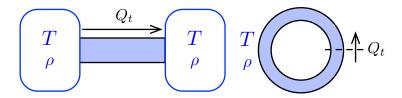
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Statistics of Q_t

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- cumulants $\langle Q_t^n \rangle_c$
- generating function $\log \langle e^{\lambda Q_t} \rangle$
- large deviation function $\operatorname{Pro}[Q_t \simeq qt] \sim e^{-t\mathcal{F}(q)}$

Non-trivial even at equilibrium :



From fluctuation-dissipation, we expect

$$\frac{\left\langle Q_t^2 \right\rangle_c}{t} \propto \frac{1}{L}$$

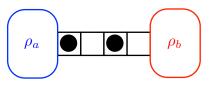
How do the $\langle Q_t^n \rangle_c$ behave?

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Diffusive systems

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Diffusive systems (symmetric exclusion processes, etc.) obey Fourier's law



Open geometry :

$$rac{1}{t}\log\left\langle e^{\lambda Q_{t}}
ight
angle \simeqrac{1}{L}\mathcal{F}(
ho_{a},
ho_{b},\lambda)$$

All cumulants decay in 1/L:

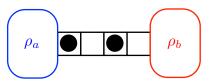
$$rac{\langle Q_t^n
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Bodineau, Derrida (2004)

Diffusive systems

Diffusive systems

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Bodineau, Derrida (2004)

 $\frac{\log \left\langle e^{\lambda Q_t} \right\rangle}{t} \simeq \frac{\alpha \lambda^2}{L} + \frac{1}{L^2} \mathcal{F}(\rho, \lambda)$

Higher cumulants in $1/L^2$

Appert, Derrida, Lecomte, van Wijland (2008)

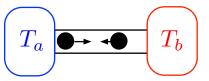
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Mechanical models

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1D momentum-conserving models have an anomalous Fourier's law



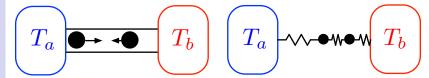
Hard particle gas :

$$rac{\langle Q_t
angle}{t} \propto rac{1}{L^{2/3}}$$

Mechanical models

Mechanical models

1D momentum-conserving models have an anomalous Fourier's law



Hard particle gas :

$$rac{\langle Q_t
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Anharmonic chain (Fermi-Pasta-Ulam ' β ') :

$$rac{\langle Q_t
angle}{t} \propto rac{1}{L^{\sim 3/5}}$$

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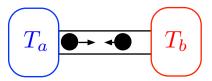
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⇒ Anomalous conduction with two different regimes Delfini, Lepri, Livi, Politi, van Beijeren, Spohn...

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Mechanical models

For the HPG, this behavior extends to the cumulants : Brunet Derrida G. (2010)



Open geometry : similar decay

$$rac{\langle Q_t^n
angle}{t} \propto rac{1}{L^{2/3}}$$

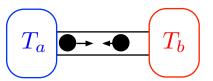
for $n \leq 4$

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Mechanical models

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Open geometry : similar decay

$$rac{\langle Q_t^n
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Ring geometry :

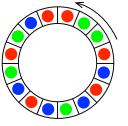
$$rac{\left\langle Q_t^2
ight
angle_c}{t} \propto rac{1}{\sqrt{L}}$$
 , $rac{\left\langle Q_t^4
ight
angle_c}{t} \propto \sqrt{L}$

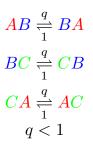
Higher cumulants seem to grow faster with L

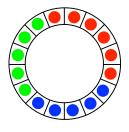
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In this talk : the ABC model

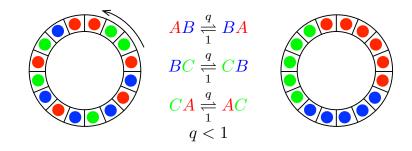






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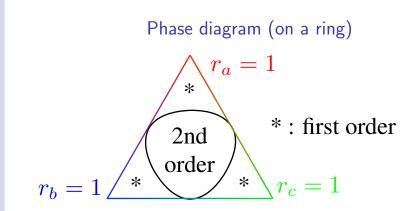
In this talk : the ABC model



- When $q = e^{-\beta/L}$: phase transition at $\beta = \beta_*(r_a, r_b, r_c)$
- Anomalous current fluctuations around β_* :

$$\frac{\langle Q_a(t)\rangle}{t} = \frac{J}{L} + \frac{\text{corr.}}{L^{3/2}} , \frac{\langle Q_a^n(t)\rangle_c}{t} \propto L^{n-5/2}$$

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- Transition can be first or second-order, depending on r_a, r_b, r_c
- If it is second-order, then it occurs at

$$\beta_* = \frac{2\pi}{\sqrt{1 - 2\sum r_i^2}}.$$

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• β_* is unknown otherwise

Anomalous current fluctuations at a

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The second-order transition

• The ABC model obeys diffusive dynamics :

 $\operatorname{Pro}[\text{site } k \text{ is of type } a] \simeq \rho_a \left(k/L, t/L^2 \right) = \rho_a(x, \tau)$

 $\begin{cases} \partial_{\tau} \rho_{a} = -\partial_{x} j_{a} & \text{(conservation)} \\ j_{a} = -\partial_{x} \rho_{a} + \beta \rho_{a} (\rho_{c} - \rho_{b}) & \text{(biased Fick's law)} \end{cases}$

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The second-order transition

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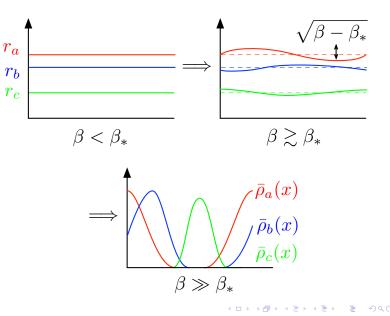
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- the constant profiles ρ_a(x, τ) = r_a are stable for β < β_{*} ⇒ homogeneous / disordered phase
- They become unstable for β ≥ β_{*} : the new stable profiles are modulated in space
 ⇒ modulated / ordered phase

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The second-order transition

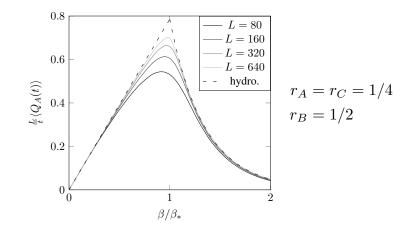


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First cumulant of the current

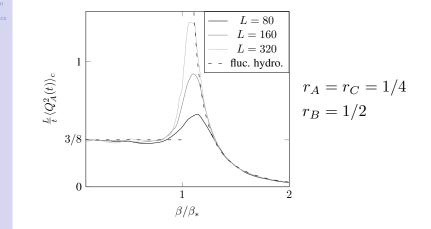


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Second cumulant

Second cumulant

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Critical regime

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- The first Fourier mode of ρ_a becomes unstable as $\beta \uparrow \beta_*$
- Assume that, for $\beta \simeq \beta_*$, this mode, R_a , varies slowly and is larger than the others :

$$\rho_a(x,\tau) = r_a + (R_a(\tau)e^{2i\pi x} + cc.) + o(R_a)$$

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$$\rho_a(x,\tau) = r_a + (R_a(\tau)e^{2i\pi x} + cc.) + o(R_a)$$

The hydrodynamics equations j_a = −∂_xρ_a + βρ_a(ρ_c − ρ_b) give the effective dynamics of R_a :

$$\frac{dR_{a}}{d\tau} = 4\pi^{2} \left[\gamma - \frac{2\Lambda}{\Delta^{2}} |R_{a}|^{2} \right] R_{a}$$

with

$$\gamma = \frac{\beta - \beta_*}{\beta_*} \qquad \Delta = 1 - 2\sum_a r_a^2 \qquad \Lambda = \sum_a r_a^2 - 2\sum_a r_a^3$$

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Fluctuation

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Constructor

Fluctuating hydrodynamics

• Fluctuations of Q_t arise from stochastic corrections to the hydrodynamics :

$$j_{a} = -\partial_{x}\rho_{a} + \beta\rho_{a}(\rho_{c} - \rho_{b}) + \frac{\eta_{a}(x,\tau)}{\sqrt{L}}$$

• $\eta_a(x,t)$: white noise $\langle \eta_a(x,\tau)\eta_b(x',\tau')\rangle = \sigma_{ab}\delta(x-x')\delta(\tau-\tau')$

with
$$\sigma_{ab} = \begin{cases} 2\rho_a(1-\rho_a) \text{ if } a = b \\ -2\rho_a\rho_b \text{ otherwise.} \end{cases}$$

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Fluctuations

Current

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Fluctuating critical regime

When adding the stochastic corrections,

$$j_{a} = -\partial_{x}\rho_{a} + \beta\rho_{a}(\rho_{c} - \rho_{b}) + \frac{\eta_{a}(x,\tau)}{\sqrt{L}}$$

the dynamics of R_a get an added complex white noise

$$\frac{dR_a}{d\tau} = 4\pi^2 \left[\gamma - \frac{2\Lambda}{\Delta^2} |R_a|^2 \right] R_a + \frac{\nu_a(\tau)}{\sqrt{L}}$$

with

$$\langle \nu_{a}(\tau)\nu_{a}^{*}(\tau')\rangle = \frac{24\pi^{2}r_{a}^{2}r_{b}r_{c}}{\Delta}\delta(\tau-\tau')$$

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Fluctuating critical regime

• Rescaling :

$$R_{a}=rac{1}{L^{1/4}}f(ar{ au})$$
 with $ar{ au}=rac{ au}{\sqrt{L}}$

 $\Rightarrow f(\bar{\tau})$ diffuses in a quartic potential :

$$\frac{df}{d\bar{\tau}} = 4\pi^2 \left(\bar{\gamma} - \frac{2\Lambda}{\Delta^2} |f|^2 \right) f + \nu(\bar{\tau}) \qquad \text{with } \bar{\gamma} = \sqrt{L} \frac{\beta - \beta_*}{\beta_*}$$

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 \Rightarrow Critical regime for $|\beta - \beta_*| \sim 1/\sqrt{L}$:

- the first Fourier mode fluctuates in $1/L^{1/4}$ (the others in $1/\sqrt{L}$)
- these fluctuations are on a slow time scale $\bar{\tau} \propto t/L^{5/2}$

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Current in the critical regime

$$Q_a(t) = L \int_0^{t/L^2} d\tau \left[\beta r_a(r_c - r_b) + \frac{2\beta}{\sqrt{L}} \frac{r_b - r_c}{r_a} |f(\bar{\tau})|^2 + (noise) \right]$$

$$\frac{\langle Q_a(t)\rangle}{t} \simeq \frac{\beta}{L} r_a(r_c - r_b) + \frac{2\beta}{L^{3/2}} \frac{r_b - r_c}{r_a} C_1(\bar{\gamma})$$
$$\frac{\langle Q_a^n(t)\rangle_c}{t} \simeq \frac{1}{L^{5/2-n}} \left[\frac{2\beta(r_b - r_c)}{r_a}\right]^n C_n(\bar{\gamma})$$

• $C_n(\bar{\gamma})$ is an *n*-point correlations integral of $|f(\bar{\tau})|^2$:

$$C_n(\bar{\gamma}) = \lim_{\bar{\tau} \to \infty} \frac{1}{\bar{\tau}} \int_0^{\bar{\tau}} d\bar{\tau}_1 ... d\bar{\tau}_n \left\langle |f(\bar{\tau}_1) ... f(\bar{\tau}_n)|^2 \right\rangle$$

(only $\mathcal{C}_1 = \left< |f|^2 \right>$ has an analytical expression)

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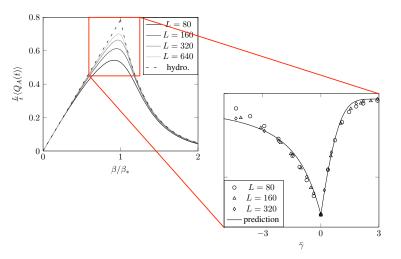
Current statistics

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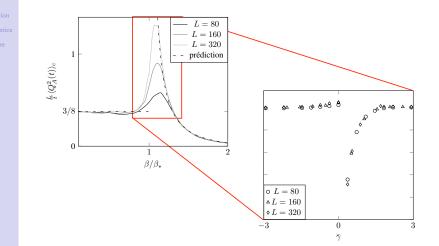
Rescaling : first cumulant



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Rescalings

Rescaling : second cumulant



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Conclusion

$$\beta \neq \beta_{*} \qquad \beta = \beta_{*}$$

$$\Lambda \neq 0 \qquad \Lambda = 0$$
Time scale
$$t/L^{2} \qquad t/L^{5/2} \qquad t/L^{8/3}$$
Cumulants
$$\frac{\langle Q_{t}^{n} \rangle_{c}}{t} \propto \frac{1}{L} \qquad \frac{\langle Q_{t}^{n} \rangle_{c}}{t} \propto L^{n-5/2} \qquad \frac{\langle Q_{t}^{n} \rangle_{c}}{t} \propto L^{4(n-2)/3}$$
Correlations
$$\langle \rho(x)\rho(y) \rangle_{c} \propto \frac{1}{L} \qquad \langle \rho(x)\rho(y) \rangle_{c} \propto \frac{1}{\sqrt{L}} \qquad \langle \rho(x)\rho(y) \rangle_{c} \propto \frac{1}{L^{1/3}}$$

Anomalous current fluctuations at a

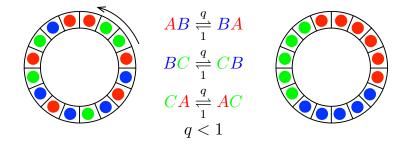
phase transition

Concl

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Why a phase transition?



• For $r_a = r_b = r_c = 1/3$, there is detailed balance with energy

$$E = \sum_{i=1}^{L-1} \sum_{j=i+1}^{L} [C_i B_j + A_i C_j + B_i A_j] \qquad A_i = \begin{cases} 1 \text{ if } i \text{ is of type } A \\ 0 \text{ otherwise} \end{cases}$$

Evans (M.) Kafri Koduvely Mukamel (1998)

- \Rightarrow effective interactions are long-ranged
 - For generic r_a, r_b, r_c , non-equilibrium steady state

Tricritical regime

First mode evolution equation

$$\frac{dR_{a}}{d\tau} = 4\pi^{2} \left[\gamma - \frac{2\Lambda}{\Delta^{2}} |R_{a}|^{2} \right] R_{a} + \frac{\nu_{a}(\tau)}{\sqrt{L}}$$

• $\Lambda = \sum r_a^2 - 2 \sum r_a^3$ vanishes on the tricritical line :

 $\left\{ \begin{array}{l} \Lambda \geq 0 \Rightarrow \text{second-order transition} \\ \Lambda < 0 \Rightarrow \text{first-order transition} \end{array} \right.$

Going to the next order yields

$$\frac{dR_{a}}{d\tau} = 4\pi^{2} \left[\gamma - \frac{|R_{a}|^{4}}{r_{a}^{2}\Delta} \right] R_{a} + \frac{\nu_{a}(\tau)}{\sqrt{L}}$$

Anomalous current fluctuations at a phase transition

Phase transition Current statistic

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Scalings and current fluctuations

• Different scaling for R_a :

$$R_{a} = rac{1}{L^{1/6}}g(ilde{ au})$$
 with $ilde{ au} = rac{ au}{L^{2/3}}$

 \Rightarrow Larger fluctuations on a slower time scale

• Faster growth of the cumulants :

$$\frac{\langle Q_{a}(t)\rangle}{t} \simeq \frac{\beta}{L} r_{a}(r_{c}-r_{b}) + \frac{2\beta}{L^{4/3}} \frac{r_{b}-r_{c}}{r_{a}} D_{1}(\bar{\gamma})$$
$$\frac{\langle Q_{a}^{n}(t)\rangle_{c}}{t} \simeq L^{4(n-2)/3} \left[\frac{2\beta(r_{b}-r_{c})}{r_{a}}\right]^{n} D_{n}(\bar{\gamma})$$

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