Generalized random cluster representation and correlation and BK inequalities

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Outline

Usual stuff

1. Gibbs distributions
2. Ising model
3. FK representation

Novelties

1. Generalized random cluster representation RCR
2. BK property
3. Foldings
4. A general result

Consequencies

1. Independence
2. BK property (in particular of antiferromagnetic Curie-Weiss)
3. Open problem for simple exclusion
4. Cluster disjoint realizations
5. An FK proof of FKG

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Part I

Usual stuff

Gibbs distributions on a finite set

- *graph*$G = (\Lambda, \mathcal{B})$,
  Λ finite set, $\mathcal{B} \subseteq \mathcal{P}(\Lambda)$,
- $\Omega = F^{\Lambda}$, F insieme finito,
- P is Gibbs for the interaction $\phi : \cup_{b \in \mathcal{B}} \Omega_b \to \mathbb{R}$
  if $P(\omega) = \frac{1}{Z} e^{\sum_{b \in \mathcal{B}} \phi(\omega_b)}$.

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Ising model

- $F = \{-1, 1\}$, $\mathcal{B} \subseteq \mathcal{B}^{(2)} = \{b \in \mathcal{B}, |b| = 2\}$,
- $P = \mu_J(\omega) = \frac{1}{Z} e^{\sum_{b=\{i,j\} \in \mathcal{B}} J\omega_i\omega_j}$, $J \in \mathbb{R}$.

FK or Random Cluster representation

- original work [Fortuin e Kasteleyn (1972)], version of [Edwards and Sokal (1988)]:
- consider $\eta \in H = \{0,1\}^{\mathcal{B}}$
  and a joint distribution Q su $\Omega \times H$, with $p \in [0,1]$

  $Q_p(\omega, \eta) = \frac{1}{Z} p^{\eta^1} (1-p)^{\eta^0} \mathbb{I}_{\omega \sim \eta}$

  where $\mathbb{I}_{\omega \sim \eta}$ indicates that $\forall \{i,j\} \in \mathcal{B}, \eta_{\{i,j\}} = 1 \Rightarrow$
  - $\omega_i = \omega_j$ in the ferromagnetic $J \geq 0$
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  and Z is a normalizing factor.

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Marginals of FK-RC Representation

$\sum_\eta Q_p(\omega, \eta) = \mu_J(\omega)$ if $p = 1 - e^{-2J}$

$\bar{\nu}(\eta) = \sum_\omega Q_p(\omega, \eta) = \frac{1}{Z} p^{\eta^0} (1-p)^{\eta^1} 2^{\mathcal{C}(\eta)}$

where $\mathcal{C}(\eta) = |\{\text{site clusters determined by active bonds } b : \eta_b = 1\}|$

Correlation and dependence

- FK-RC Representation can be used to bound spin-spin correlation by random cluster percolation:
  $Corr(\omega_i, \omega_j) = \bar{\nu}(i \leftrightarrow j) = \bar{\nu}(i \text{ is connected to } j \text{ using active bonds })$

In essence Random Cluster connectivity measures dependence.

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Part II

Novelties

Generalized RCR

Idea

- Rewrite the FK-RCR of the Ising model
  $Q_p(\omega,\eta) = \frac{1}{Z} p^{\eta^1}(1-p)^{\eta^0}\mathbb{I}_{\omega\sim\eta} = \frac{1}{Z}\nu_p(\eta)\prod_{b\in\mathcal{B}}\mathbb{I}_{\omega_b\in\eta_b}$

  where
  $\eta \in H = \prod_{b\in\mathcal{B}}(\mathcal{P}(\Omega_b))$
  and ν_p is a probability on H.

Generalized RCR

- ν is a $\mathcal{B}$-RCR of P if it is a probability on H and $P(\omega) = \sum_\eta \frac{1}{Z}\nu(\eta)\prod_{b\in\mathcal{B}}\mathbb{I}_{\omega_b\in\eta_b}$.
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Remarks on the generalized RCR

Elementary properties

- Given P on a finite Λ there exists at least one $\mathcal{B}$ such that there is a $\mathcal{B}$-RCR of P;
- given P and $\mathcal{B}$ there might be no, one or many $\mathcal{B}$-RCR's of P;
- Say that the RCR is Bernoulli iff ν is a product measure.
  Theorem: P has a Bernoulli $\mathcal{B}$-RCR $\iff$ there exists ϕ such that P is Gibbs in $(\Lambda, \mathcal{B})$ with interaction ϕ.

Active hyperbonds in the RCR and marginal

One can still define the marginal $\bar{\nu}$ of ν on H (but attention: ν is Bernoulli).
Also, hyperbond $b \in \mathcal{B}$ is inactive in η if $\eta_b = \Omega_b$,
otherwise b is **active**.

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Inutility of the generalized RCR

Unfortunately, connection by active hyper bonds is scarcely related to spin dependence:

- it might be $\bar{\nu}(\Lambda_A \leftrightarrow \Lambda_B) = 0$ and still A and B dependent under P;
- it might be $\bar{\nu}(i \leftrightarrow j) > 0$ and still ω_i and ω_j independent under P (even for ν Bernoulli);
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Negative dependence

Restrict to $F = \{-1, 1\}$.

Negative association NA

- NA: $A, B \uparrow$, if the supports of A and B are disjoint, i.e. $\exists \Lambda_1, \Lambda_2 \subseteq \Lambda, \Lambda_1 \cap \Lambda_2 = \emptyset, A \in \Omega_{\Lambda_1}, B \in \Omega_{\Lambda_2}$, then $P(A \cap B) \leq P(A)P(B)$
- NA is disccused in Pemantle (1991); an important sufficient condition is in Borcea, Branden and Liggett (2008) which in invariant under the simple exclusion process showing that the simple exclusion is NA

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- NA: $A, B \uparrow$, if the supports of A and B are disjoint, i.e. $\exists \Lambda_1, \Lambda_2 \subseteq \Lambda, \Lambda_1 \cap \Lambda_2 = \emptyset, A \in \Omega_{\Lambda_1}, B \in \Omega_{\Lambda_2}$, then $P(A \cap B) \leq P(A)P(B)$
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BK and R properties

Here is a stronger property than NA.

BK and R properties

- For two events A and B let

$$A \square B = \{\omega \in \Omega | \text{ there exist } \Lambda_1(\omega), \Lambda_2(\omega) \subseteq \Lambda, \\ \Lambda_1(\omega) \cap \Lambda_2(\omega) = \emptyset, \omega|_{\Lambda_1(\omega)} \subseteq A, \omega|_{\Lambda_2(\omega)} \subseteq B\}$$

- Increasing event $A \uparrow$ with respect to semiorder in Ω: $\omega \in A, \omega' \geq \omega \to \omega' \in A$
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Known results

- van den Berg, Kesten (1985): P Bernoulli is BK.
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Foldings

Reimer uses the foldings of a probability. Let $F = \{-1, 1\}$ for semplicity. Then the folding is obtained by taking two configurations ω, ω', fixing the region M where they agree and have value $\alpha = \alpha_M$ and letting the remaining part fluctuate randomly (subject to the constraint that configurations disagree).
In more formal terms:

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$$\begin{aligned} W_{K,\alpha} &= \{(\omega, \omega') \in \Omega \times \Omega | \\ &\qquad \omega|_K = \omega'|_K = \alpha, \omega|_{K^c} = -\omega'|_{K^c} = \bar{\omega}'|_{K^c}\}, \end{aligned}$$

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A general result

- Theorem ([van den Berg, G. , PTRF (2012) to appear]). Given P and two events A and B. If $A\square^* B$ is the event that A and B are realized using certain disjoint sets, and for each folding there is a symmetric RCR such that the above sets are not connected by active hyperbonds, then $P(A\square^* B) \leq P(A)P(B)$.

Foldings can be defined for more than two valued variables, and the theorem still holds

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- Given P, with RCR $\nu_{K,\alpha}$ for the folding $\tilde{P}^{K,\alpha}$, A, B and γ, if for every $\omega \in A\square_\gamma B$ and every K, α, there is $(M, N) \in \gamma(A, B, \omega)$ such that (*) $M \cap K^c$ and $N \cap K^c$ are not connected by active hyperbonds in $(\Lambda, \mathcal{B}(K^c)_\eta)$ for every $\eta \sim \omega|_{K^c}$
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- Suppose that A and B are based on two disjoint sets (like ω_i and ω_j), then $A\square^* B = A \cap B$;
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A graphical condition for the BK property

Let $F = \{-1, 1\}$.

- If $A \uparrow$ and $B \uparrow$ are increasing, then they are identified by sets Λ_A such that $\omega_{\Lambda_A} \equiv 1$
- then it would be sufficient to have or each folding
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- One can then fix a distribution P and look for the RCR's with the above properties.

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Antiferromagnetic Curie-Weiss model

- It is the antiferromagnetic Ising model ($J \leq 0$) on the complete graph $\mathcal{B} = \mathcal{B}^{(2)} = \{b \in \mathcal{B}, |b| = 2\}$ [Kac (1968)]

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3 - Open problem for simple exclusion

- One could wonder if the simple exclusion process is also BK (besides NA)

  it would be sufficient to show that each folding of the distribution (at time t or ∞)

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4 - Cluster disjoint realizations

- Our starting point was actually the following problem:
  take a configuration $\omega \in \{-1, 1\}^{\Lambda}$ of the Ising model and divide it into clusters
  then use the clusters to recognize A and B
  form the event $A \Box_{cl} B$
  the sets Λ_A and Λ_B only touch with opposite signs
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Repeated foldings and FKG theorem

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- $\tilde{P}^{K,\alpha}$ is a distribution on Ω_{K^c}, so one can consider his foldings $K_2 \subset K^c$, $\alpha_2 \in \Omega_{K^c \setminus K_2}$ and so o.
- One gets a tree of foldings
  although the main result applies to just one step, a weaker result like FKG applies to repeated steps

Repeated foldings and FKG theorem

Repeated foldings

- $\tilde{P}^{K,\alpha}$ is a distribution on Ω_{K^c}, so one can consider his foldings $K_2 \subset K^c$, $\alpha_2 \in \Omega_{K^c \setminus K_2}$ and so o.
- One gets a tree of foldings
  although the main result applies to just one step, a weaker result like FKG applies to repeated steps

5 - A RCR proof of FKG

- Here is a (sketch) of a proof of FKG theorem using generalized RCR.
- If P is FKG, the so is every folding $\tilde{P}^{K,\alpha}$
- It is easy to prove FKG theorem in the leaves of the tree of foldings
- Going backwards P is positively associated.

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