Interacting random maps in $\mathbb{Z}$
A construction

Inés Armendáriz, Pablo A. Ferrari,
Pablo Groisman, Florencia Leonardi

Buenos Aires, São Paulo

Florence, August 2012

Weighted Bijections

Finite interval $\Lambda \subset \mathbb{Z}$

$\sigma_\Lambda : \Lambda \to \Lambda$ a bijection (permutation)

$\alpha > 0$ real parameter.

Hamiltonian:

$$H_\Lambda(\sigma_\Lambda) = \sum_{x \in \Lambda} (x - \sigma_\Lambda(x))^2$$

Specification in Λ: $\sigma : \mathbb{Z} \to \mathbb{Z}$ with $\sigma(x) = x$ for $x \notin \Lambda$

$$\mu_\Lambda(\sigma) = \frac{1}{Z_\Lambda} \exp(-\alpha H_\Lambda(\sigma))$$

where Z_Λ is the normalizing constant.

Thermodynamic limit

Gibbs measure μ as a limit

$$\lim_{\Lambda \nearrow \mathbb{Z}} \mu_\Lambda \quad ?$$

More dimensions? Other point sets (Poisson process, etc)?

Motivation in **Gandolfo-Ruiz-Ueltschi 2007**:

Gaussian weights: Feynman-Kac representation of Bose gas. Toth model.

α is **proportional to the temperature** of the system.

Model introduced by Feynman (1953), Kikuchi (1954), Kikuchi, Denman, Schreiber (1960), Fichtner (1991), Betz and Ueltschi (2008, '09, '11).

Our motivation: **Biskup Richthammer** (talk at EBP 2010).

Cycle of a bijection σ

sequence of distinct sites $(x_0, \ldots, x_n)$ such that

$$\sigma(x_i) = x_{i+1}; \quad \sigma(x_n) = x_0$$

σ is identified by $\{\gamma : \gamma \text{ is a cycle of } \sigma\}$.

Problems:

"There should be no long cycles for α large, i.e. when sites are heavily discouraged from jumping to a neighbor."

"Cycles should increase in size when α decreases."

"The main question is whether a transition occurs for some value $\alpha_c > 0$, below which a fraction of the sites find themselves in infinitely long cycles. "

Ground states

σ' is a **local perturbation of the bijection** σ if

$$\{x \in \mathbb{Z} : \sigma'(x) \neq \sigma(x)\} \qquad \text{is finite}$$

σ is a **ground state** if for any local perturbation σ' of σ,

$$H(\sigma') - H(\sigma) > 0$$

(computable because the number of differences is finite).

For $n \in \mathbb{Z}$ let η^n be the bijection

$$\eta^n(x) = x + n, \qquad x \in \mathbb{Z}$$

Lemma: $\{\eta^n,\ n \in \mathbb{Z}\}$ are **all** ground states for H.

When $n \neq 0$ these ground states have cycles of infinite length.

Reason: In a ground state there cannot be crossings: picture!

$$(x-\eta(x))^2+(y-\eta(y))^2 \;<\; (x-\eta(y))^2+(y-\eta(x))^2$$

when $x<y$ and $\eta(x)<\eta(y)$.

"Gibbs measures must be perturbations of these ground states."

Gibbs formalism

Hamiltonian: For σ bijection and Λ finite:

$$H_\Lambda(\sigma) = \sum_{x\in\Lambda}(x - \sigma(x))^2.$$

Boundary conditions: Let Λ finite, η bijection and define

$$B(\Lambda,\eta) := \{\sigma \text{ bijection } : \sigma(x) = \eta(x),\ x \in \Lambda^c\}$$

Specifications: probability measure $\mu_{\Lambda,\eta}$ on $B(\Lambda,\eta)$:

$$\mu_{\Lambda,\eta}(\sigma) := \frac{1}{Z_{\Lambda,\eta}} \exp\big(-\alpha H_\Lambda(\sigma)\big)\,\mathbf{1}\{\sigma \in B(\Lambda,\eta)\},$$

Gibbs measures

- μ on $\{$bijections$\}$ is a **Gibbs measure** for the family $(\mu_{\Lambda,\eta})$ if

$$\mu(\sigma \text{ occurs in } \Lambda \,|\, \eta \text{ occurs in } \Lambda^c) = \mu_{\Lambda,\eta}(\sigma)$$

for μ-almost all η and all finite Λ.

- $\mathcal{G}^\alpha$: (convex) set of Gibbs for temperature α.
- Extremal Gibbs characterize $\mathcal{G}^\alpha$.
- Ergodic are Gibbs.

Flow of a bijection η Define

$$F_x^+(\eta) := \sum_{y \le x} \mathbf{1}\{\eta(y) > x\}$$

$$F_x^-(\eta) := \sum_{y > x} \mathbf{1}\{\eta(y) \le x\}$$

(Net) Flow: $F(\eta) := F^+(\eta) - F^-(\eta)$

- Flow is constant in x. Denote $F^+(\eta), F^-(\eta)$.
- *n*-Gibbs: $\mathcal{G}_n^\alpha :=$ extremal Gibbs measures with flow n.

Results of Biskup and Richthammer

For all $\alpha > 0$,

1. Gibbs measures have finite flow:

If $\mu \in \mathcal{G}^\alpha$, then $\mu(F^+ < \infty, F^- < \infty) = 1$.

2. n-Gibbs measures have $|n|$ infinite cycles:

For $n \in \mathbb{Z}$, if $\mu \in \mathcal{G}_n^\alpha$, then there are n infinite cycles μ-a.s.

3. For each n there is only one n-Gibbs measure.

4. $\{$extremal Gibbs$\} = \{n$-Gibbs$\}$

5. Thermodynamic limit. For all η with flow n

$$\lim_{\Lambda \nearrow \mathbb{Z}} \mu_{\Lambda,\eta} \text{ exists and is } n\text{-Gibbs}$$

.

Comments on Biskup-Richthammer:

BR also prove **non–existence at temperature 0**. This relies on the nonexistence of a uniform probability distribution on an infinite countable set.

The **uniqueness** at positive temperature is obtained by a **cutblock argument** used in the proof of **2**.

The **existence** at positive temperature is based on a **tightness argument** and proving that tightness preserves the flow.

“Shift” operator

Observation: **It suffices to study the 0-Gibbs measure.**

Given $n \in \mathbb{Z}$, let the shift operator $\theta^n : \sigma \mapsto \theta^n \sigma$ be defined by

$$(\theta^n \sigma)(x) = \sigma(x) + n$$

So that $\theta^n \mathrm{Id} = \eta^n$, where Id is the identity.

Lemma.

$$\mu(\theta^n)^{-1} \text{ is } n\text{-Gibbs} \quad \Leftrightarrow \quad \mu \text{ is 0-Gibbs}$$

Explicit construction of the 0-Gibbs measure for α large.

Theorem (AFGL)

There exists $\alpha_c > 0$ such that if $\alpha > \alpha_c$:

- *Explicit construction of a coupling $(\sigma_\Lambda, \Lambda \subset \mathbb{Z})$, such that*

for finite Λ, $\sigma_\Lambda \sim \mu_{\Lambda,\mathrm{Id}}$ on $B(\Lambda, \mathrm{Id})$, the specification on Λ with identity bondary conditions, and

$$\lim_{\Lambda \nearrow \mathbb{Z}} \sigma_\Lambda = \sigma_{\mathbb{Z}} \qquad a.s.$$

- *The law of $\sigma_{\mathbb{Z}}$ is the only 0-Gibbs measure.*
- *All cycles in $\eta_{\mathbb{Z}}$ are finite.*
- *Results export to n-Gibbs using the shift operator θ^n.*

$n = 0$: (Non-trivial) Cycles Let $k \geq 2$,

- Cycle $\gamma = (x_0, \dots, x_n, x_{k+1})$, $x_0 = x_{k+1}$, $\sigma(x_i) = x_{i+1}$.
- $o(\gamma) = \min\{x_0, \dots, x_n\}$ is the *origin* of γ.
- $|\gamma| = k + 1$ number of distinct sites used by γ.
- The *weight* of γ

$$w(\gamma) = \alpha \sum_{i=0}^{k} (x_i - x_{i+1})^2$$

- σ identified with $\Gamma(\sigma) := \{$ non-trivial disjoint cycles of $\sigma\}$.
- $\gamma \sim \gamma'$, *compatible* if they use disjoint sets of sites.

Loss network: birth and death of cycles

- Current state: set of cycles $\Gamma = \Gamma(\sigma)$

- At rate $e^{-w(\gamma)}$ propose cycle γ. If it is compatible with present cycles, then γ is *born*. If not, γ is "lost".

- Each present cycle *dies* at rate 1.

- For finite region Λ, finite state Markov process with generator:

$$L_\Lambda f(\Gamma) = \sum_{\gamma\subset\Lambda} e^{-w(\gamma)} \mathbf{1}\{\gamma \sim \Gamma\} \Big[f(\Gamma\cup\{\gamma\}) - f(\Gamma)\Big] + \sum_{\gamma\in\Gamma} \Big[f(\Gamma\setminus\{\gamma\}) - f(\Gamma)\Big]$$

- Call Γ_t^Λ the resulting process and η_t^Λ the process induced on the set of bijections of Z.

- $\mu_{\Lambda,\mathrm{Id}}$ is reversible for η_t^Λ. Trivial for finite Λ.

Stationary version and thermodynamical limit

Theorem. *(Fernández, F, Garcia 2001)*

If

$$m(\alpha) := \sum_{\gamma : o(\gamma)=0} |\gamma| \, e^{-w(\gamma)} < 1$$

then:

- *A stationary version of the loss network $(\eta_t^\Lambda,\ t \in \mathbb{R})$ can be constructed for all Λ finite or infinite. Call μ the marginal law of $\eta_t = \eta_t^{\mathbb{Z}}$.*

- *For finite Λ the time-marginal law of η_t^Λ is $\mu_{\Lambda,\mathrm{Id}}$.*

- *Spatial limits: Simultaneous coupling of $(\eta_t^\Lambda,\ t \in \mathbb{R},\ \Lambda \subset \mathbb{Z})$ such that for all $x \in \mathbb{Z}$,*

$$\eta_t^\Lambda(x) \to \eta_t(x) \text{ almost surely as } \Lambda \nearrow \mathbb{Z}.$$

Stationary version and thermodynamical limit

- *Consequence:* *Thermodynamic limit:* $\mu_{\Lambda,\mathrm{Id}} \to \mu$ *weakly.*
- μ, *the law of* η_t, *is* 0*-Gibbs.*

- *Time limits, coupling from the past:* *If* ζ *has finite cycles, then coupling* $(\eta_{[-t,0]}, \eta^{\zeta}_{[-t,0]})$ *such that for all* x,

$$\lim_{t\to\infty} \eta^{\zeta}_{[-t,0]}(x) \to \eta_{[-t,0]}(x)$$

almost surely.

- *Consequences: 1)* η^{ζ}_t *converges weakly to* μ.

2) μ *is the only invariant measure for* η_t

Sketch of proof

Free process.

- Free birth and death process of cycles (superpositions allowed).

- The invariant measure for the free process is a product of Poisson: number of cycles of type γ is Poisson with mean $e^{-w(\gamma)}$.

- Stationary construction (time in $\mathbb{R}$) of the free process.

- Space and time limits for the free process are trivial.

- Free process generates random set of space-time cylinders.

Graphic construction of the Loss network (exclusion interaction)

- Works as the free process but cycles incompatible with present cycles cannot appear.

- Construction coupled to free construction forward in time

Use the free cylinders but erase (loose) those incompatible with present cylinders. Proceed iteratively.

- Clan of ancestors of a cylinder C: cylinders born in the past that may influence the birth of C.

- If clan of ancestors born after time 0 is finite, then the process exists. This happens if $m(\alpha) < \infty$.

- If clan of ancestors is finite, stationary construction ok.

- This happens if $m(\alpha) < 1$.

- In this case space and time limits hold exponentially fast.

- Invariant measure for this process in finite Λ is $\mu_{\Lambda,\mathrm{Id}}$. This is the time marginal of η_t^Λ.

- The time marginal of the infinite volume stationary process η_t is Gibbs.

A particle system question

- Suppose that $m(\alpha) = \infty$.

- In this case the loss network is not well defined, at least starting with the empty configuration. The rate of birth of a cycle with origin at 0 is infinite.

- But the 0-Gibbs measure is “invariant” for this non existent dynamics (BR).

- Does it exist the stationary dynamics in this case?

Other open questions

- Are there infinite cycles in more than 2 dimensions? Beltz-Uelchi have arguments supporting this conjecture.

- Can one describe all the ground states 2 or more dimensions?

Larger dimensions: $d \geq 2$

- No backwards percolation approach works for large α in $\mathbb{Z}^d$. We can construct the (unique) 0-Gibbs state as a gas of weighted cycles interacting by exclusion.

- Open: to establish the ground states in $d \geq 2$.

- Extremal measures? The lemma above no longer holds.

- Other hamiltonians? $(x - \pi(x))^p$ for instance? Construction works for high α in the same way.

Construction of μ^k from μ^0

- Finite perturbations of the ground state η^n can be obtained in terms of perturbations of η^0.

- Shift operator: $\theta^n \sigma$ as the operator

$$\theta^n \sigma(x) = \sigma(x) + n$$

Notice that $(\theta^n)^{-1} = \theta^{-n}$.

Let L_n be the dynamics defined by

$$L_n \theta^n \sigma = \theta^n L \sigma$$

Map to a configuration with flow 0, apply the zero dynamics, map back to the configuration with flow n.

Lemma. *If $\mu \in \mathcal{G}_n$, then μ is reversible for L_n.*

Proof. Configurations σ, σ'.

Denote by $\Lambda = \{x \in \mathbb{Z} \,:\, \sigma'(x) \neq \sigma(x)\}$.

$\theta^n \sigma'(x) = \theta^n \sigma(x)$ for any $x \in \Lambda^c$, therefore

$$\begin{aligned}
H(\theta^n \sigma') - H(\theta^n \sigma) &= \sum_{x \in \Lambda} [x - \sigma'(x) - n]^2 - \sum_{x \in \Lambda} [x - \sigma(x) - n]^2 \\
&= \sum_{x \in \Lambda} [x - \sigma'(x)]^2 - \sum_{x \in \Lambda} [x - \sigma(x)]^2 \\
&\quad - 2n \left[\sum_{x \in \Lambda} \sigma'(x) - \sum_{x \in \Lambda} \sigma(x) \right] \\
&= H(\sigma') - H(\sigma)
\end{aligned}$$

because $\sigma'(\Lambda) = \sigma(\Lambda)$ and $\sum_{x \in \Lambda} \sigma'(x) = \sum_{x \in \Lambda} \sigma(x)$.