

CURRENT FLUCTUATIONS IN DIFFUSIVE SYSTEMS AND PHASE TRANSITIONS

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COLLABORATORS

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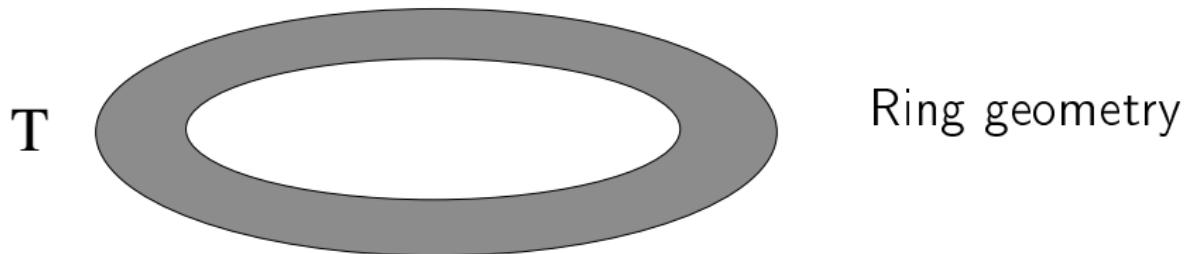
A. Gerschenfeld

Firenze 2012

Current fluctuations in non-equilibrium steady states

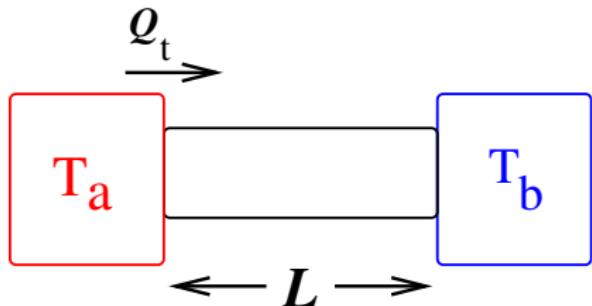


Current fluctuations in non-equilibrium steady states



NON EQUILIBRIUM STEADY STATE

HEAT

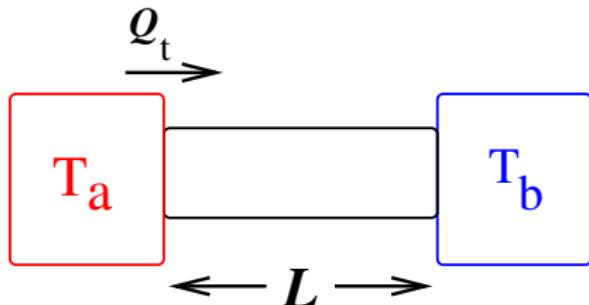


Fourier's law

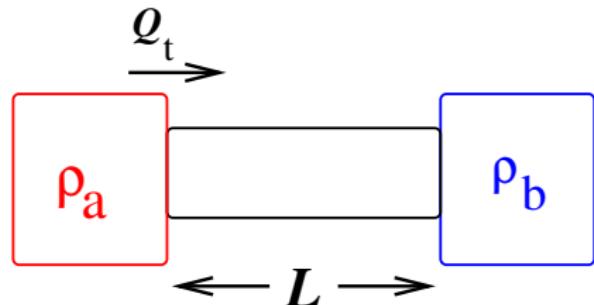
$$\frac{\langle Q_t \rangle}{t} \sim \frac{D(T_a, T_b)}{L}$$

NON EQUILIBRIUM STEADY STATE

HEAT



PARTICLES



Fourier's law

$$\frac{\langle Q_t \rangle}{t} \sim \frac{D(T_a, T_b)}{L}$$

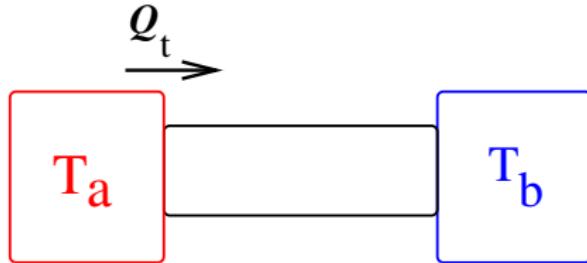
Fick's law

$$\frac{\langle Q_t \rangle}{t} \sim \frac{D(\rho_a, \rho_b)}{L}$$

Distribution of Q_t

$P(Q_t) ?$

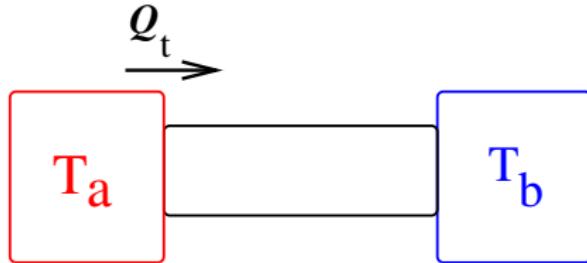
CURRENT FLUCTUATIONS AND LARGE DEVIATIONS



Q_t energy transferred during a long time t

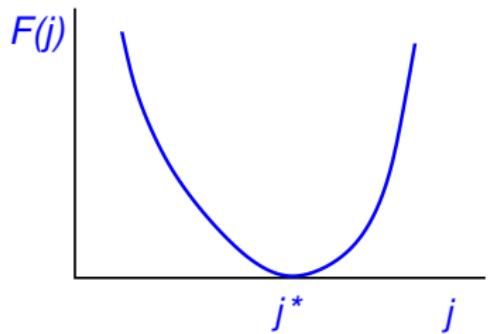
$$\text{Pro} \left(\frac{Q_t}{t} = j \right) \sim \exp[-t F(j)]$$

CURRENT FLUCTUATIONS AND LARGE DEVIATIONS



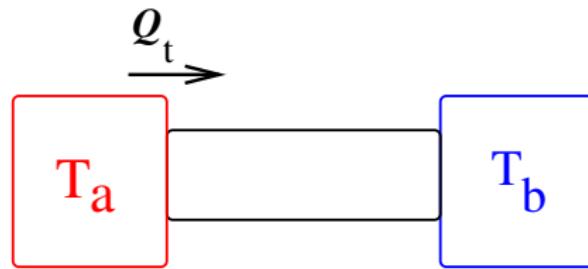
Q_t energy transferred during a long time t

$$\text{Pro} \left(\frac{Q_t}{t} = j \right) \sim \exp[-t F(j)]$$



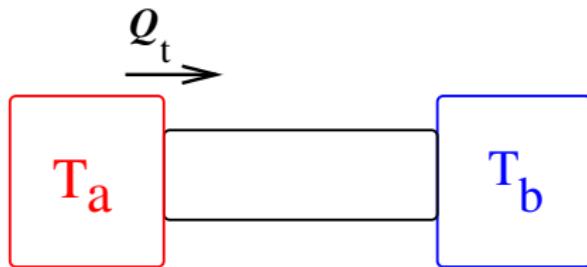
Expansion of $F(j)$ near j^* gives all cumulants of Q_t

CURRENT FLUCTUATIONS AND LARGE DEVIATIONS



$$\langle \exp [\lambda Q_t] \rangle \sim \exp [t \mu(\lambda)]$$

CURRENT FLUCTUATIONS AND LARGE DEVIATIONS



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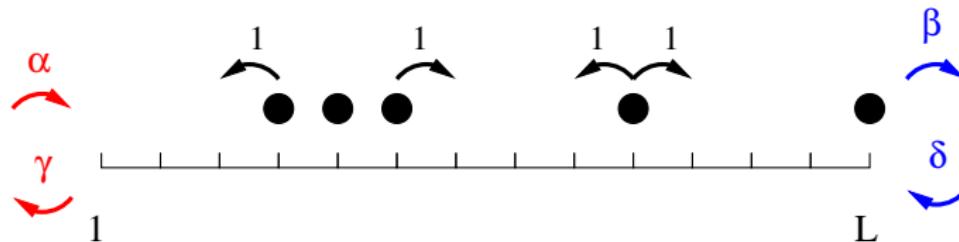
$$\text{Pro} \left(\frac{Q_t}{t} = j \right) \sim \exp[-t F(j)]$$

Legendre transform

$$\mu(\lambda) = \max_j [\mu j - F(j)]$$

EXCLUSION PROCESSES

SSEP (Symmetric simple exclusion process)



$$\rho_a = \frac{\alpha}{\alpha + \gamma} ,$$

$$\rho_b = \frac{\delta}{\beta + \delta}$$

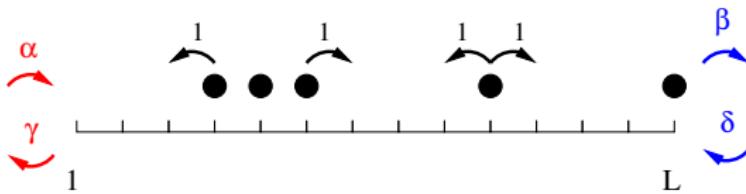
$$\text{Pro}\left(\frac{Q_t}{t} = j\right) \sim \exp[-t F(j)]$$

?

$$\langle \exp [\lambda Q_t] \rangle \sim \exp [t \mu(\lambda)]$$

TWO APPROACHES

SSEP (Symmetric simple exclusion process)



Microscopic

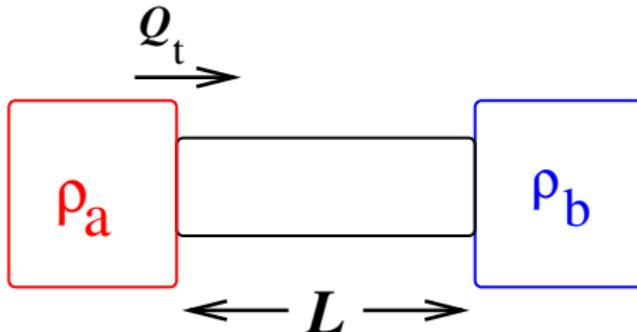
Bethe ansatz, Perturbation theory, Computer,...

Macroscopic

$$i = Lx, \quad t = L^2\tau$$

$$\text{Pro}(\{\rho(x, \tau), j(x, \tau)\}) \sim \exp \left[-L \int_0^{T/L^2} dt \int_0^1 dx \frac{[j + \rho']^2}{4\rho(1-\rho)} \right]$$

MICROSCOPIC APPROACH



$$\langle e^{\lambda Q_t} \rangle \sim e^{t \mu(\lambda)} ?$$

The evolution (Markov process)

$$\frac{dP(\mathcal{C})}{dt} = \sum_{\mathcal{C}'} W(\mathcal{C}, \mathcal{C}') P(\mathcal{C}') - W(\mathcal{C}', \mathcal{C}) P(\mathcal{C})$$

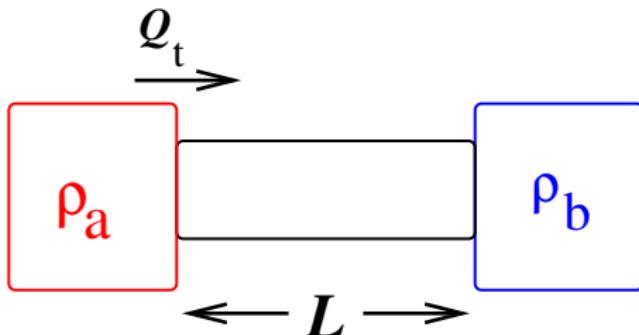
One can decompose

$$W(\mathcal{C}, \mathcal{C}') = W_1(\mathcal{C}, \mathcal{C}') + W_0(\mathcal{C}, \mathcal{C}') + W_{-1}(\mathcal{C}, \mathcal{C}')$$

$W_q(\mathcal{C}, \mathcal{C}')$ represents a jump $\mathcal{C}' \rightarrow \mathcal{C}$ with $Q_t \rightarrow Q_t + q$

$$\mu(\lambda) = \text{largest eigenvalue of } e^{\lambda W_1} + W_0 + e^{-\lambda W_{-1}}$$

PHASE TRANSITION



$$\langle e^{\lambda Q_t} \rangle \sim e^{t \mu_L(\lambda)}$$

For a finite system $\mu_L(\lambda)$ is analytic

Non analyticity in the large L limit \equiv phase transition

Finite size effects

MICROSCOPIC APPROACHES:

perturbation theory, Bethe ansatz, Matrix method, ...

J. de Gier, F.H.L. Essler,

Bethe ansatz solution of the asymmetric exclusion process with open boundaries

Phys. Rev. Lett. 95 (2005) 240601

D. Simon,

Construction of a coordinate Bethe ansatz for the asymmetric simple exclusion process with open boundaries

J. Stat. Mech. (2009) P07017

A. Lazarescu, K. Mallick,

An exact formula for the statistics of the current in the TASEP with open boundaries

J. Phys. A: Math. Theor. 44 (2011) 315001

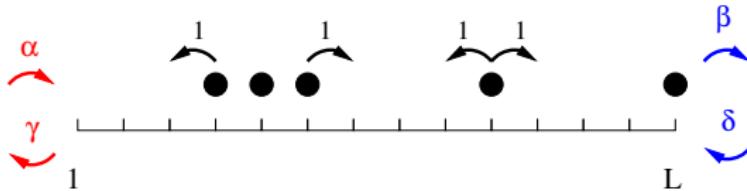
M. Gorissen, A. Lazarescu, K. Mallick, C. Vanderzande,

Exact current statistics of the ASEP with open boundaries

arXiv:1207.6879

SSEP (Symmetric simple exclusion process)

D. Doucet Roche 2004

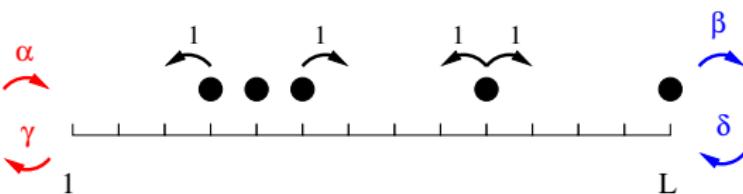


$$\rho_a = \frac{\alpha}{\alpha + \gamma}, \quad \rho_b = \frac{\delta}{\beta + \delta}$$

$$\lim_{t \rightarrow \infty} \frac{\langle Q(t) \rangle}{t} \simeq \frac{1}{L} [\rho_a - \rho_b] \quad \text{Fick's law}$$

$$\lim_{t \rightarrow \infty} \frac{\langle Q^2(t) \rangle_c}{t} \simeq \frac{1}{L} \left[\rho_a + \rho_b - \frac{2(\rho_a^2 + \rho_a \rho_b + \rho_b^2)}{3} \right]$$

$$\lim_{t \rightarrow \infty} \frac{\langle Q^3(t) \rangle_c}{t} \simeq \frac{1}{L} (\rho_a - \rho_b) \left[1 - 2(\rho_a + \rho_b) + \frac{16\rho_a^2 + 28\rho_a \rho_b + 16\rho_b^2}{15} \right]$$



$$\rho_a = \frac{\alpha}{\alpha + \gamma} ,$$

$$\rho_b = \frac{\delta}{\beta + \delta}$$

D. Doucet Roche 2004

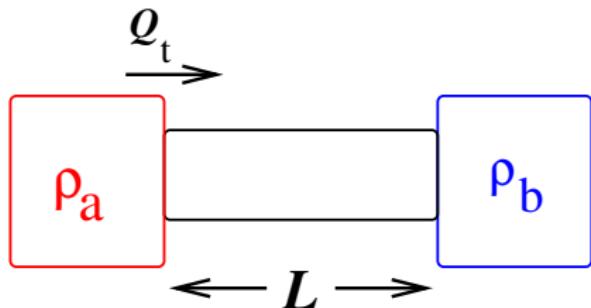
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$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{\langle Q^4(t) \rangle_c}{t} \simeq & \frac{1}{L} \left[\rho_a + \rho_b - \frac{2(7\rho_a^2 + \rho_a \rho_b + 7\rho_b^2)}{3} \right. \\ & \left. + \frac{32\rho_a^3 + 8\rho_a^2 \rho_b + 8\rho_a \rho_b^2 + 32\rho_b^3}{5} - \frac{96\rho_a^4 + 64\rho_a^3 \rho_b - 40\rho_a^2 \rho_b^2 + 64\rho_a \rho_b^3 + 96\rho_b^4}{35} \right] \end{aligned}$$

DIFFUSIVE SYSTEMS



Diffusive system

► For $\rho_a - \rho_b$ small: $\frac{\langle Q_t \rangle}{t} = \frac{D(\rho)(\rho_a - \rho_b)}{L}$

► $\rho_a = \rho_b = \rho$: $\frac{\langle Q_t^2 \rangle}{t} = \frac{\sigma(\rho)}{L}$

SSEP: $D = 1$ and $\sigma = 2\rho(1 - \rho)$

MACROSCOPIC FLUCTUATION THEORY FOR DIFFUSIVE SYSTEMS

Kipnis Olla Varadhan 89

Onsager Machlup theory for non equilibrium

Spohn 91

$$\frac{d\rho}{dt} = -\frac{dj}{dx} \quad (\text{conservation law})$$

Bertini De Sole Gabrielli
Jona-Lasinio Landim 2001 →

Evolution of a profile $\rho(x, t)$ for $0 \leq t \leq T$

$\text{Pro}(\{\rho(x, t), j(x, t)\})$

$$\exp \left[-L \int_0^{T/L^2} dt \int_0^1 dx \frac{[j(x, t) + \rho'(x, t)D(\rho(x, t))]^2}{2\sigma(\rho(x, t))} \right]$$

$\hat{\hat{}}$

$$j(x, t) = -\rho'(x, t)D(\rho(x, t)) + \frac{1}{\sqrt{L}}\eta(x, t)$$

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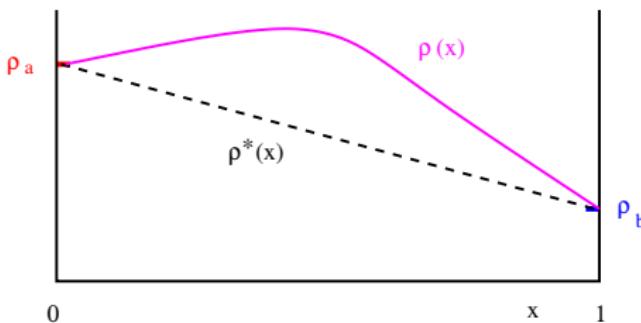
with the white noise $\langle \eta(x, t)\eta(x', t') \rangle = 2\sigma(\rho(x, t))\delta(x - x')\delta(t - t')$

VARIATIONAL PRINCIPLE assuming that $j(x, t) = j$

Bodineau D. 2004

Large deviation function $P(Q_t/t) \sim \exp[-tF_L(j)]$

$$F_L(j, \rho_a, \rho_b) = \frac{1}{L} \min_{\rho(x)} \int_0^1 dx \frac{[j L + \rho'(x) D(\rho(x))]^2}{2\sigma(\rho(x))}$$



Satisfies the fluctuation theorem

$$F(j) - F(-j) = j \int_{\rho_a}^{\rho_b} \frac{D(\rho)}{\sigma(\rho)} d\rho$$

Gallavotti Cohen 1995

Evans Searls 1994

.... Kurchan 1998

Lebowitz Spohn 1999

CONSEQUENCES

$$\langle e^{\lambda Q_t} \rangle \sim e^{t \mu(\lambda)}$$

$$\mu(\lambda, \rho_a, \rho_b) = -\frac{K}{L} \left[\int_{\rho_b}^{\rho_a} \frac{D(\rho) d\rho}{\sqrt{1 + 2K\sigma(\rho)}} \right]^2$$

$$\lambda = \int_{\rho_b}^{\rho_a} d\rho \frac{D(\rho)}{\sigma(\rho)} \left[\frac{1}{\sqrt{1 + 2K\sigma(\rho)}} - 1 \right]$$

Cumulants

$$\frac{\langle Q_t \rangle_c}{t} = \frac{1}{L} \mathcal{I}_1$$

$$\frac{\langle Q_t^2 \rangle_c}{t} = \frac{1}{L} \frac{\mathcal{I}_2}{\mathcal{I}_1}$$

$$\frac{\langle Q_t^3 \rangle_c}{t} = \frac{1}{L} \frac{3 (\mathcal{I}_3 \mathcal{I}_1 - \mathcal{I}_2^2)}{\mathcal{I}_1^3}$$

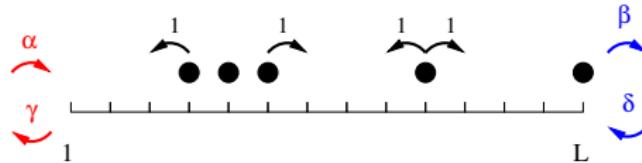
$$\frac{\langle Q_t^4 \rangle_c}{t} = \frac{1}{L} \frac{3 (5 \mathcal{I}_4 \mathcal{I}_1^2 - 14 \mathcal{I}_1 \mathcal{I}_2 \mathcal{I}_3 + 9 \mathcal{I}_2^3)}{\mathcal{I}_1^5}$$

where

$$\mathcal{I}_n = \int_{\rho_b}^{\rho_a} D(\rho) \sigma(\rho)^{n-1} d\rho$$

For the SSEP $D(\rho) = 1$ and $\sigma(\rho) = 2\rho(1-\rho)$

CURRENT FLUCTUATIONS IN THE SSEP



$$\rho_a = \frac{\alpha}{\alpha + \gamma}$$

$$\rho_b = \frac{\delta}{\beta + \delta}$$

$$\langle e^{Q_t} \rangle \sim e^{t \mu(\lambda)}$$

For large L and λ small

$$\mu(\lambda, \alpha, \gamma, \beta, \delta) = \frac{1}{L} R(\omega)$$

where $\omega = 1 - [1 - (e^\lambda - 1)\rho_a][1 - (1 - e^{-\lambda})\rho_b]$

Result: $R(\omega) = [\log(\sqrt{1 + \omega} + \sqrt{\omega})]^2$

TRUE VARIATIONAL PRINCIPLE

Bertini De Sole Gabrielli
Jona-Lasinio Landim 2005

$$F(j) = \frac{1}{L} \lim_{T \rightarrow \infty} \min_{\rho(x,t), j(x,t)} \frac{1}{T} \int_0^T dt \int_0^1 dx \frac{[j(x,t) + \rho'(x,t)D(\rho(x,t))]^2}{2\sigma(\rho(x,t))}$$

with $\frac{d\rho}{dt} = -\frac{dj}{dx}$ (conservation), $\rho_t(0) = \rho_a$, $\rho_t(1) = \rho_b$ and

$$\int_T j_t(x) dt$$

- ▶ Sufficient condition for the optimal profile to be time independent
- ▶ Dynamical phase transition

the optimal $\rho_t(x)$ starts to become time dependent

$N = L\rho_0$ on a ring of L sites

$$j(x, t) = -\rho'(x, t) + \nu\sigma(\rho(x, t)) + \eta(x, t)$$

with the white noise $\langle \eta(x, t)\eta(x', t') \rangle = 2\sigma(\rho(x, t))\delta(x - x')\delta(t - t')$

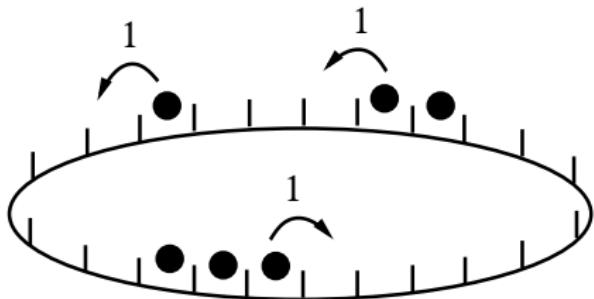
Optimal profile

$$\rho(x, t) = \rho_0 \text{ is flat} \qquad \qquad \rho(x, t) = g(x - vt)$$



$$\mu(\lambda) = \mu_{\text{flat}} \simeq \frac{\lambda(\lambda + 2\nu)\sigma(\rho_0)}{L} \qquad \qquad \mu(\lambda) > \mu_{\text{flat}}$$

TASEP (totally asymmetric exclusion process)



N particles L sites

$$\rho = \frac{N}{L}$$

$$\langle Q \rangle = t\rho(1 - \rho)$$

D. Lebowitz 1998; Appert D. 1999

$$\langle e^{\lambda Q_t} \rangle \sim e^{t \mu(\lambda)}$$

$$\mu(\lambda) - \frac{\lambda \langle Q \rangle}{t} = \sqrt{\frac{\rho(1 - \rho)}{L^3}} \mathcal{G} \left(\lambda \sqrt{L\rho(1 - \rho)} \right)$$

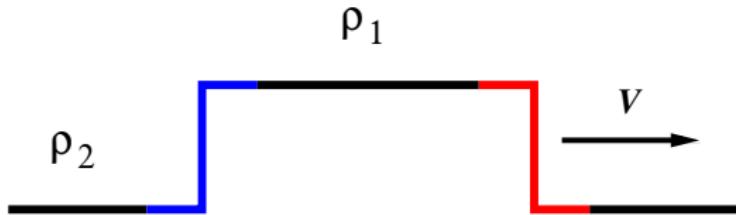
For $\lambda < 0$

$$\mu(\lambda) = -\frac{(1 - e^{\lambda\rho})(1 - e^{\lambda(1-\rho)})}{1 - e^\lambda}$$

TASEP:

shocks and anti shocks on the line

Jensen Varadhan 2000



$$v = 1 - \rho_1 - \rho_2$$

$$\text{Pro(anti shock)} = \exp[-t \Phi(\rho_1, \rho_2)]$$

where

$$\Phi(\rho_1, \rho_2) = \rho_2 - \rho_1 - \rho_1 \rho_2 \log \frac{\rho_2}{\rho_1} - (1 - \rho_1)(1 - \rho_2) \log \frac{1 - \rho_2}{1 - \rho_1}$$

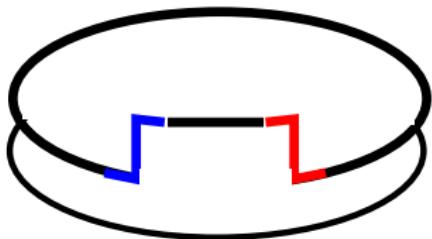
LARGE DEVIATION FUNCTION

Bodineau D. 2005

$$\langle e^{\lambda Q_t} \rangle \sim e^{t \mu(\lambda)}$$

For $\lambda < 0$

$$\mu(\lambda) = -\frac{(1 - e^{\lambda\rho})(1 - e^{\lambda(1-\rho)})}{1 - e^\lambda}$$



One can recover this expression by

$$\mu(\lambda) = \max_{\rho_1, \rho_2} \{ [\lambda[y\rho_1(1 - \rho_1) + (1 - y)\rho_2(1 - \rho_2)] - \Phi(\rho_1, \rho_2)\}$$

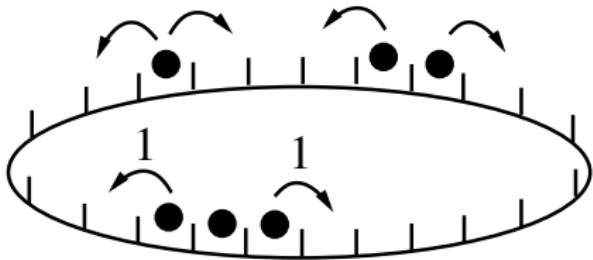
where $\rho = y\rho_1 + (1 - y)\rho_2$

SSEP ON A RING finite size effects

Appert D Lecomte Van Wijland 2008

N particles
 L sites

$$\rho = \frac{N}{L}$$



Q_t flux through a bond during time t

CUMULANTS OF THE CURRENT FOR THE SSEP ON A RING

N particles and L sites

$$\sigma(\rho) = 2\rho(1 - \rho) = \frac{2N(L-N)}{L^2}$$

$$\frac{\langle Q^2 \rangle_c}{t} = \frac{\sigma}{L-1}$$

$$\frac{\langle Q^4 \rangle_c}{t} = \frac{\sigma^2}{2(L-1)^2}$$

$$\frac{\langle Q^6 \rangle_c}{t} = -\frac{(L^2-L+2)\sigma^3 - 2(L-1)\sigma^2}{4(L-1)^3(L-2)}$$

$$\frac{\langle Q^8 \rangle_c}{t} = \frac{(10L^4-2L^3+27L^2-15L+18)\sigma^4 - 4(L-1)(11L^2-L+12)\sigma^3 + 48(L-1)^2\sigma^2}{24(L-1)^4(L-2)(L-3)}$$

$$\boxed{\frac{\langle Q^2 \rangle_c}{t} = \frac{\sigma}{L}}$$

Gaussian

+ Fick's law

$$\boxed{\frac{\langle Q^{2n} \rangle_c}{t} \sim \frac{\sigma^n}{L^2}}$$

for $n \geq 2$

UNIVERSAL CUMULANTS OF THE CURRENT

$$\frac{\langle Q^2 \rangle_c}{t} = \frac{\sigma}{L}$$

Gaussian

$$\frac{\langle Q^4 \rangle_c}{t} \simeq \frac{\sigma^2}{2L^2}, \quad \frac{\langle Q^6 \rangle_c}{t} \simeq -\frac{\sigma^3}{4L^2}, \quad \frac{\langle Q^8 \rangle_c}{t} \simeq \frac{5\sigma^4}{12L^2}$$

Universal

UNIVERSAL CUMULANTS OF THE CURRENT

$$\frac{\langle Q^2 \rangle_c}{t} = \frac{\sigma}{L}$$

Gaussian

$$\frac{\langle Q^4 \rangle_c}{t} \simeq \frac{\sigma^2}{2L^2}, \quad \frac{\langle Q^6 \rangle_c}{t} \simeq -\frac{\sigma^3}{4L^2}, \quad \frac{\langle Q^8 \rangle_c}{t} \simeq \frac{5\sigma^4}{12L^2}$$

Universal

$$\langle e^{\lambda Q_t} \rangle \sim e^{t \mu(\lambda)}$$

with

$$\mu(\lambda) - \frac{\lambda^2}{2} \frac{\langle Q^2 \rangle_c}{t} = \frac{1}{L^2} \mathcal{F}\left(-\frac{\sigma \lambda^2}{4}\right)$$

$$\mathcal{F}(u) = -4 \sum_{n \geq 1} \left[n\pi \sqrt{n^2\pi^2 - 2u} - n^2\pi^2 + u \right] = \frac{1}{3}u^2 + \frac{1}{45}u^3 + \frac{1}{378}u^4 + \dots$$

\mathcal{F} universal

Singularity as $u \rightarrow \frac{\pi^2}{2}$

FLUCTUATING HYDRODYNAMICS

Gaussian expansion of the **macroscopic fluctuation theory** around a constant current and a flat profile.

$$\rho(x, t) = \rho + \sum_{k, \omega} k [a_{k, \omega} e^{i\omega\tau + ikx} + a_{k, \omega}^* e^{-i\omega t - ikx}]$$

$$j = j_0 - \omega [a_{k, \omega} e^{i\omega\tau + ikx} + a_{k, \omega}^* e^{-i\omega t - ikx}] .$$

Gaussian fluctuations

$$\text{Pro}(Q_t = j_0 t, \{a_{k, \omega}\}) \sim$$

$$\exp \left[-\frac{j_0^2}{2\sigma} \frac{t}{L} - \frac{t}{L} \sum_{\omega, k} |a_{k, \omega}|^2 \left(\frac{(\sigma\omega + j_0\sigma'k)^2}{\sigma^3} + \frac{D^2 k^4}{\sigma} - \frac{j_0^2 \sigma'' k^2}{2\sigma^2} \right) \right]$$

- ▶ Integrate over the fluctuations
- ▶ Sum over the discrete modes k

RESULTS FOR A GENERAL DIFFUSIVE SYSTEM

$$\langle e^{\lambda Q_t} \rangle \sim e^{t \mu(\lambda)}$$

$$\mu(\lambda) - \frac{\lambda^2 \langle Q^2 \rangle}{2t} = \frac{1}{L^2} D \mathcal{F} \left(\frac{\sigma \sigma''}{16D^2} \lambda^2 \right)$$

$$\mathcal{F}(u) = -4 \sum_{n \geq 1} \left[n\pi \sqrt{n^2\pi^2 - 2u} - n^2\pi^2 + u \right] = \frac{1}{3}u^2 + \frac{1}{45}u^3 + \frac{1}{378}u^4 + \dots$$

- ▶ Phase transition as $u \rightarrow \pi^2/2$
- ▶ For $n \geq 2$

$$\frac{\langle Q^{2n} \rangle_c}{t} \sim \frac{1}{L^2} \frac{(2n)! B_{2n-2}}{n! (n-1)!} D \left(\frac{\sigma \sigma''}{8D^2} \right)^n$$

B_n Bernoulli numbers

CONCLUSION

Phase transition in the large deviation function of the current

flat profile → traveling wave profile

finite size effects in the flat phase

OPEN QUESTIONS

More complicated time dependent profiles

Open geometry

RELATED QUESTIONS

Phase transition in the large deviation function of the density

L. Bertini, A. De Sole, D. Gabrielli, G. Jona Lasinio, C. Landim, Lagrangian phase transitions in nonequilibrium thermodynamic systems
J. Stat. Mech. L11001 (2010)

G. Bunin, Y. Kafri, D. Podolsky, Title: Non differentiable large-deviation functionals in boundary-driven diffusive systems

GENERATING FUNCTION OF THE CURRENT

$$\langle e^{\lambda Q_t} \rangle \sim e^{t \mu(\lambda)}$$

?

$$\mu(\lambda) = \text{largest eigenvalue of } e^\lambda W_1 + W_0 + e^{-\lambda} W_{-1}$$

The evolution

$$\frac{dP(\mathcal{C}, Q)}{dt} = \sum_q \sum_{\mathcal{C}'} W_q(\mathcal{C}, \mathcal{C}') P(\mathcal{C}', Q - q) - W_q(\mathcal{C}', \mathcal{C}) P(\mathcal{C}, Q)$$

$W_q(\mathcal{C}, \mathcal{C}')$ represents a jump $\mathcal{C}' \rightarrow \mathcal{C}$ with $Q_t \rightarrow Q_t + q$

GENERATING FUNCTION OF THE CURRENT

$$\langle e^{\lambda Q_t} \rangle \sim e^{t \mu(\lambda)} ?$$

$$\mu(\lambda) = \text{largest eigenvalue of } e^\lambda W_1 + W_0 + e^{-\lambda} W_{-1}$$

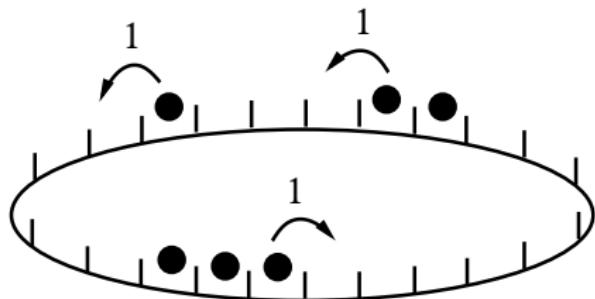
The evolution

$$\sum_Q e^{\lambda Q} \left(\frac{dP(\mathcal{C}, Q)}{dt} = \sum_q \sum_{\mathcal{C}'} W_q(\mathcal{C}, \mathcal{C}') P(\mathcal{C}', Q - q) - W_q(\mathcal{C}', \mathcal{C}) P(\mathcal{C}, Q) \right)$$

$$\text{Define } R(\mathcal{C}, \lambda) = \sum_Q P(\mathcal{C}, Q) e^{\lambda Q}$$

$$\frac{dR(\mathcal{C}, \lambda)}{dt} = \sum_q \sum_{\mathcal{C}'} e^{\lambda q} W_q(\mathcal{C}, \mathcal{C}') R(\mathcal{C}', \lambda) - W_q(\mathcal{C}', \mathcal{C}) R(\mathcal{C}, \lambda)$$

TASEP (totally asymmetric exclusion process)- PHASE TRANSITION



N particles L sites

$$\rho = \frac{N}{L}$$

$$\langle Q \rangle = t\rho(1 - \rho)$$

D. Lebowitz 1998; Appert D. 1999

$$\langle e^{\lambda Q_t} \rangle \sim e^{t \mu(\lambda)} \quad \mu(\lambda) - \frac{\lambda \langle Q \rangle}{t} = \sqrt{\frac{\rho(1 - \rho)}{L^3}} \mathcal{G} \left(\lambda \sqrt{L\rho(1 - \rho)} \right)$$

$$\mathcal{G}(x) = -\frac{1}{\sqrt{2\pi}} \sum_{n \geq 1} z^n n^{-3/2} \quad ; \quad x = -\frac{1}{\sqrt{2\pi}} \sum_{n \geq 1} z^n n^{-5/2}$$