

A RANDOM WALK ON TOP OF A CONTACT PROCESS

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§ RANDOM WALK IN RANDOM ENVIRONMENT

RWRE has been a highly active area of research since the early 1970's. The object of interest is a random walk in discrete/continuous space-time for which the transition probabilities/rates are random themselves.

In $d = 1$ our understanding of RWRE is fairly complete. In $d \geq 2$ many beautiful results have been obtained, but there are still some very hard open problems.

What makes RWRE particularly interesting is that new phenomena occur due to slow-down in rare space pockets.

§ RANDOM WALK IN DYNAMIC RANDOM ENVIRONMENT

RWDRE is a variant of RWRE where the random transition probabilities/rates **evolve with time**.

The state of the art is rather modest. In fact, the area has started to develop properly only since 2000. Presently there are some 40 papers in the literature.

Is the behavior of RWDRE similar to that of RWRE or not? In what way does the dynamics affect the slow-down in rare space pockets?

Three classes of DRE have been considered so far:

1. Independent in time: globally updated each unit of time.
2. Independent in space: locally updated according to independent single-site Markov chains.
3. Dependent in space and time.

Class 3 is the most challenging. Often additional assumptions are needed, such as:

- fast decay of space-time correlations.
- weak interaction: perturbative regime.

§ THIS TALK

- DRE: one-dimensional contact process in equilibrium.
- RW: nearest-neighbor random walk with asymmetric jump rates.

CHALLENGE: Contact process has slow decay of space-time correlations.

The goal is to derive a strong law of large numbers, a functional central limit theorem and a large deviation principle under appropriate assumptions on the model parameters.

DRE: Let

$$\xi = (\xi_t)_{t \geq 0}, \quad \xi_t = \{\xi_t(x) : x \in \mathbb{Z}\}$$

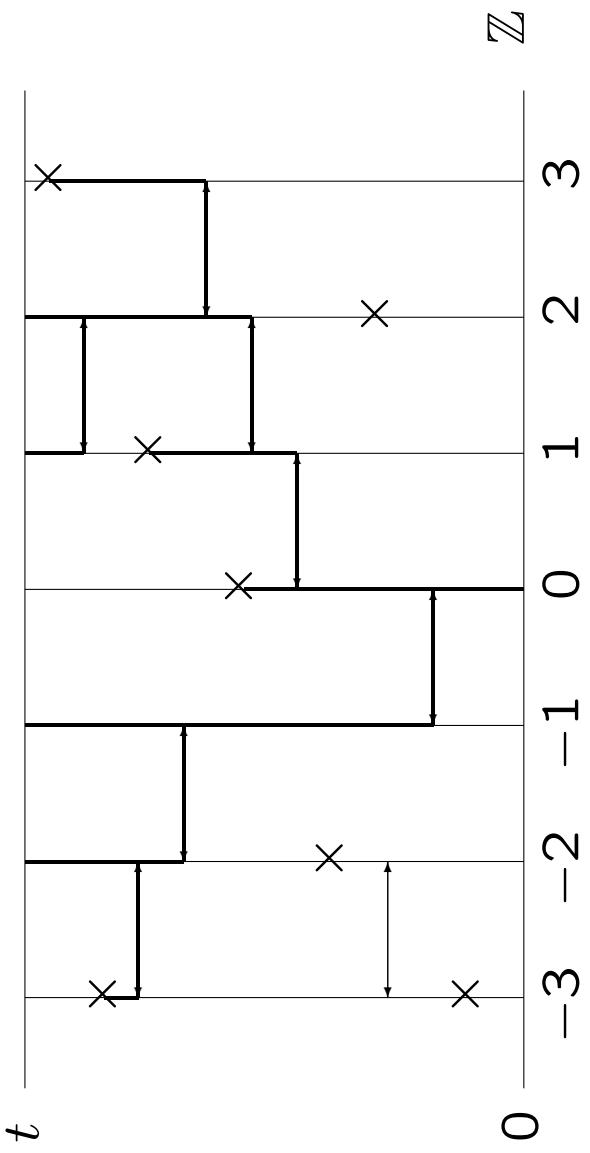
be the one-dimensional contact process, where $\xi_t(x) = 0/1$ means that site x is healthy/infected at time t .

Infected sites become healthy at rate 1, and at rate $\lambda \in (0, \infty)$ transmit their infection to sites on the right/left.

There exists a **critical infection rate** $\lambda_c \in (0, \infty)$ such that any finite infection dies out when $\lambda \leq \lambda_c$, but spreads at a positive speed when $\lambda > \lambda_c$. In the latter case there is a unique equilibrium distribution ν_λ with

$$\rho_\lambda = \nu_\lambda(\xi(0, 0)) \in (0, 1).$$

Liggett 1985



Graphical representation. Crosses are **healing events** (rate 1), arrows are **infection events** (rate λ). The thick lines cover the region that is infected when the initial configuration has a **single infection** at the origin.

RW: Given ξ , let

$$W = (W_t)_{t \geq 0}$$

be the random walk with transition rates

$$x \rightarrow x + 1 \text{ at rate } \alpha_1 \xi_t(x) + \alpha_0 [1 - \xi_t(x)],$$

$$x \rightarrow x - 1 \text{ at rate } \beta_1 \xi_t(x) + \beta_0 [1 - \xi_t(x)],$$

where $\alpha_1 + \beta_1 = \alpha_0 + \beta_0 \in (0, \infty)$.



§ LLN

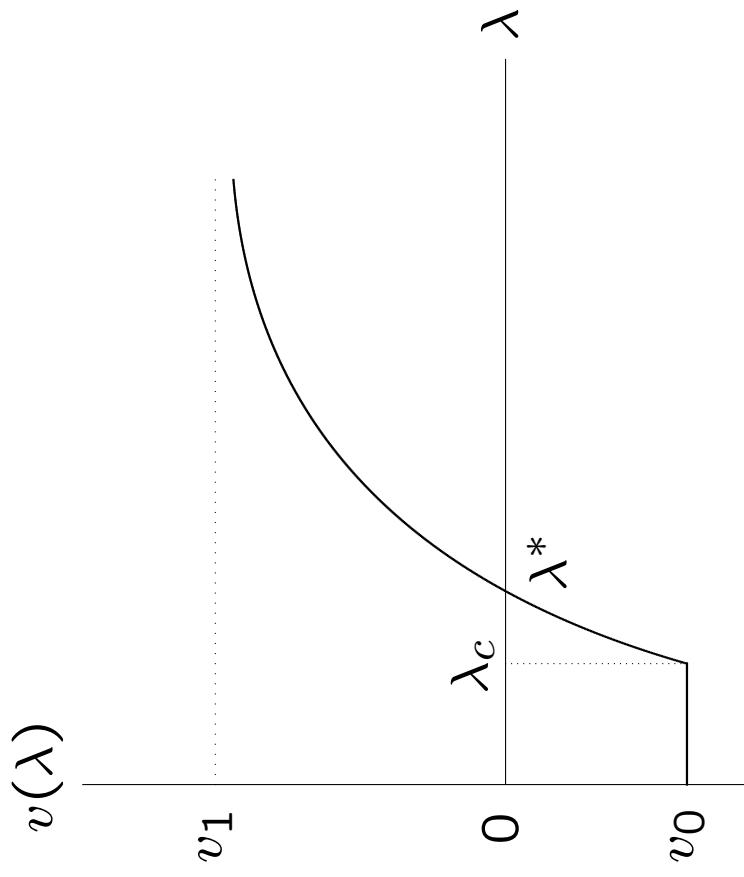
Let \mathbb{P}_{ν_λ} denote the joint law of (ξ, W) when ξ starts from ν_λ . Let $v_i = \alpha_i - \beta_i$, $i = 0, 1$. W.l.o.g. assume $v_1 > v_0$.

THEOREM:

(a) For every $\lambda \in (\lambda_c, \infty)$ there exists a $v(\lambda) \in [v_0, v_1]$ such that

$$\lim_{t \rightarrow \infty} t^{-1} W_t = v(\lambda) \quad \mathbb{P}_{\nu_\lambda}\text{-a.s. and in } L^p, p \geq 1.$$

(b) The function $\lambda \mapsto v(\lambda)$ is right-continuous and non-decreasing on (λ_c, ∞) , with $v(\lambda) \in (v_0, v_1)$ for all $\lambda \in (\lambda_c, \infty)$ and $\lim_{\lambda \rightarrow \infty} v(\lambda) = v_1$.



Qualitative plot of $\lambda \mapsto v(\lambda)$ when $0 \in (v_0, v_1)$.

It is natural to expect that $\lambda \mapsto v(\lambda)$ is continuous and strictly increasing on (λ_c, ∞) with $\lim_{\lambda \downarrow \lambda_c} v(\lambda) = v_0$. We have been unable to prove this.

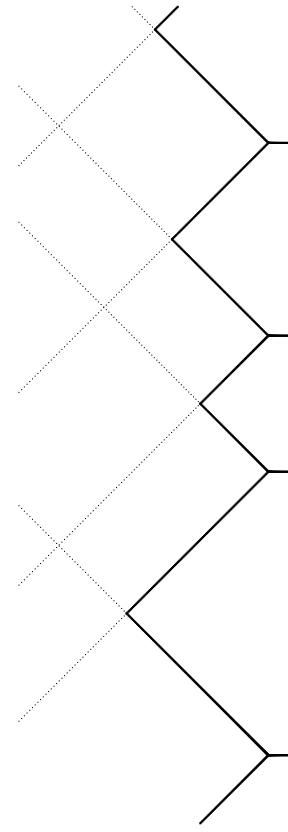
If $\lambda \in (0, \lambda_c)$, then ξ_t is known to be healthy on an interval that grows exponentially fast in t , and so it is trivial to deduce that W satisfies the LLN with $v(\lambda) = v_0$.

QUESTIONS:

- Is $\lambda \mapsto v(\lambda)$ concave on (λ_c, ∞) and Lipschitz at λ_c ?
- We know that W is transient when $v(\lambda) \neq 0$. Is W recurrent when $v(\lambda) = 0$?

The proof of the LLN proceeds by:

- Using subadditivity to prove the LLN when ξ starts from δ_1 (= all sites infected).
- Transferring the LLN by
 - coupling two copies of ξ , starting from ν_λ and δ_1 , inside space-time cones with tips at large times;
 - noting that W eventually gets trapped inside the union of the space-time cones “generated by single infections”.



§ CLT + LDP

Suppose, in addition, that $\lambda \in (\lambda_W, \infty)$ with

$$\lambda_W = \inf \{ \lambda \in (\lambda_c, \infty) : |v_0| \vee |v_1| < \iota(\lambda) \},$$

where $\lambda \mapsto \iota(\lambda)$ is the **infection propagation speed**, which is known to be continuous, strictly positive and strictly increasing on (λ_c, ∞) , with

$$\lim_{\lambda \downarrow \lambda_c} \iota(\lambda) = 0, \quad \lim_{\lambda \rightarrow \infty} \iota(\lambda) = \infty.$$

THEOREM:

(a) For every $\lambda \in (\lambda_W, \infty)$ there exists a $\sigma(\lambda) \in (0, \infty)$ such that, under \mathbb{P}_{ν_λ} ,

$$\left(\frac{W_{nt} - v(\lambda)nt}{\sigma(\lambda)\sqrt{n}} \right)_{t \geq 0} \xrightarrow{w} (B_t)_{t \geq 0} \quad \text{as } n \rightarrow \infty,$$

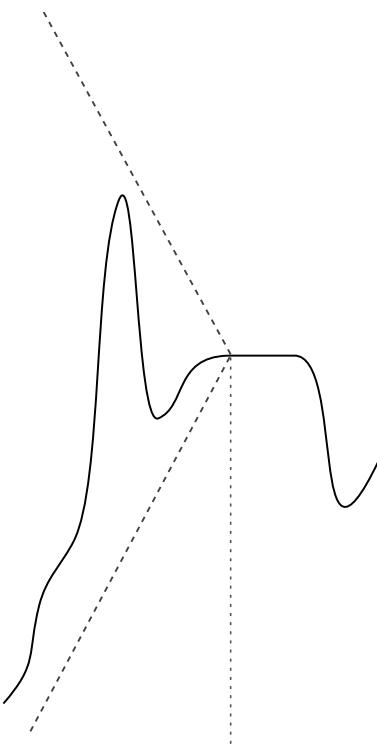
where the limit is standard Brownian motion and \xrightarrow{w} denotes weak convergence in path space.

(b) The functions $\lambda \mapsto v(\lambda)$ and $\lambda \mapsto \sigma(\lambda)$ are continuous on (λ_W, ∞) .

(c) For every $\lambda \in (\lambda_W, \infty)$ under \mathbb{P}_{ν_λ} the law of $t^{-1}W_t$ satisfies the large deviation principle on \mathbb{R} with rate t and rate function I that is convex and has a unique zero at $v(\lambda)$.

We expect the restriction $\lambda \in (\lambda_W, \infty)$ to be redundant, but we are **unable** to prove this.

The role of the restriction is that it allows us to construct regeneration times, i.e., times at which the random walk forgets its past. Infections spread so fast that the random walk eventually gets **trapped** forever inside the infection cluster generated by a single infection.



§ RELATED WORK

- DRE cone-mixing, RW elliptic.
Luca Avena, FdH, Frank Redig
- DRE cone-mixing, RW non-elliptic.
FdH, Renato dos Santos, Vladas Sidoravicius
- DRE Exclusion Process, RW non-nested.
Luca Avena, Renato dos Santos, Florian Völlerling
- DRE Exclusion Process, RW slow/fast jump rate.
Luca Avena, Philip Thomann
Luca Avena
- DRE Independent Random Walks, RW elliptic.
FdH, Harry Kesten, Vladas Sidoravicius
Marcelo Hilario, FdH, Vladas Sidoravicius