Villa Finally

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Long Range Last Passage Percolation

Francis Comets

Université Paris-Diderot

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Joint work with Jeremy Quastel and Alejandro Ramirez

Particle System

System of N particle evolving in $\mathbb{R}$:

$$X_i(t+1) = \max_{1 \le j \le N} \left\{ X_j(t) + \xi_{i,j}(t+1) \right\}, \tag{1}$$

with $\{\xi_{i,j}(s) : 1 \le i, j \le N, s \ge 1\}$ i.i.d.

• mutation / selection
• long-range Oriented Last Passage Percolation with N sites in the transverse direction:

$$X_i(t) = \max \left\{ X_{j_0}(0) + \sum_{s=1}^{t} \xi_{j_s, j_{s-1}}(s); 1 \le j_s \le N \; \forall s = 0, \dots t-1, j_t = i \right\}, \tag{2}$$

Brunet and Derrida [2004]

Outline

Fixed N

$$N-\text{vector} \quad X(t)=(X_i(t), 1\leq i\leq N)$$

Particle remain grouped under the dynamics.

Front location $\Phi(X(t))$ describing the "mean position":
$\Phi:\mathbb{R}^N\to\mathbb{R}$ symmetric, increasing, $\Phi(x+r\mathbf{1})=r+\Phi(x)$.
Examples: $\Phi(x)=\max_{i=1,\ldots,N} x_i$, $\Phi(x)=$ median, arithmetic mean, or

$$\Phi(x)=\ln\sum_{1\leq i\leq N} e^{x_i},\ldots$$

Proposition

$X(t)-\Phi(X(t))\mathbf{1}$ Markov chain in $\mathbb{R}^N$, ergodic (Doeblin, regeneration,...)

Fixed N, large t

If $\xi \in L^1$, the following limits exists a.s.

$$v_N = \lim_{t \to \infty} t^{-1} \max\{X_i(t); 1 \leq i \leq N\} = \lim_{t \to \infty} t^{-1} \Phi(X(t))$$

$v_N =$ Speed of the front.
Moreover, if $\xi \in L^2$,

$$t^{-1/2} \big(\max\{X_i(t); 1 \leq i \leq N\} - v_N t \big)$$

converges in law as $t \to \infty$ to a Gaussian r.v. with variance $\sigma_N^2 \in (0, \infty)$.

Outline

Magics of Gumbel

Gumbel law $G(a, \lambda)$

$$\mathbb{P}(\xi \leq x) = \exp\left(- e^{-\lambda(x-a)}\right), \qquad x \in \mathbb{R}.$$

Theorem (Brunet and Derrida 2004)

Assume ξ are Gumbel. Then, for $\Phi(x) = \lambda^{-1} \ln \sum_{i=1}^N \exp \lambda x_i$, the sequence $(\Phi(X(t)); t \geq 0)$ is a random walk, with increments

$$\Upsilon = a + \lambda^{-1} \ln \left(\sum_{i=1}^{N} \mathcal{E}_i^{-1} \right)$$

where the $\mathcal{E}_i$ are i.i.d. exponential of parameter 1.

$$v_N = a + \lambda^{-1} E \ln \left(\sum_{i=1}^{N} \mathcal{E}_i^{-1} \right), \qquad \sigma_N^2 = \lambda^{-2} \mathrm{Var} \left(\ln \sum_{i=1}^{N} \mathcal{E}_i^{-1} \right).$$

Fluctuations in Gumbel case

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Fluctuations in Gumbel case

- Brunet and Derrida conclude that, for fixed N, the fluctuations of the front location for $t \to \infty$ are Gaussian.
- As $N \to \infty$,

$$\mathcal{S}^{(N)} := \frac{\sum_{i=1}^{N} \mathcal{E}_i^{-1} - b_N}{N} \xrightarrow{\text{law}} \mathcal{S},$$

where $b_N = NE(\mathcal{E}^{-1}; \mathcal{E}^{-1} < N)$, and $\mathcal{S}$ is the totally asymmetric stable law of index $\alpha = 1$ (Cauchy), with characteristic function

$$\begin{aligned} Ee^{iu\mathcal{S}} &= \exp\left\{ iCu - \frac{\pi}{2}|u|\{1 + i\frac{2}{\pi}\mathrm{sign}(u)\ln|u|\}\right\} \\ &=: \exp\Psi_C(u), \end{aligned}$$

for some real constant C and

$$\ln b_N = \ln N + \ln\ln N - \frac{\gamma}{\ln N} + \mathcal{O}(\frac{1}{\ln^2 N})$$

So:

$$\Upsilon_N = \ln \sum_{i=1}^{N} \mathcal{E}_i^{-1} = \ln(b_N + N\mathcal{S}^{(N)}) = \ln b_N + \frac{N\mathcal{S}^{(N)}}{b_N} + \dots$$

Fluctuations in Gumbel case

Theorem (CQR 2012)

Assume $\xi_{i,j}(t) \sim \mathrm{G}(0,1)$. Then, for all sequences $m_N \to \infty$ as $N \to \infty$,

$$\frac{\Phi(X([m_N\tau])) - \beta_N m_N \tau}{m_N / \ln N} \xrightarrow{\text{law}} \mathcal{S}(\tau)$$

in the Skorohod topology with $\mathcal{S}(\cdot)$ a totally asymmetric Cauchy process with Lévy exponent ψ_C where $\beta_N = \ln b_N + N b_N^{-1} \ln m_N$, and
$\ln b_N = \ln N + \ln \ln N - \frac{\gamma}{\ln N} + \mathcal{O}(\frac{1}{\ln^2 N})$

- Scaling limit of the front location as the number N of particles diverges is Cauchy.
- Result also holds when time is not speeded-up ($m_N = 1$).
- Fluctuations become macroscopic only when $\ln N = o(m_N)$.

Outline

Front profile

$$U_N(t,x) = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{X_i(t)>x}, \qquad x \in \mathbb{R}$$

Front profile moving like a traveling wave in reaction-diffusion (with discretization and stochastic effects).
Should be examined around the front location $\Phi(X(t))$ or $\Phi(X(t-1))$.

From Brunet and Derrida 2004: For $\xi \sim \mathrm{G}(0,1)$,

- $$X(t) = \Phi(X(t-1))\mathbf{1} + G(t)$$
  where $G(t)$ is a N-sample of $\mathrm{G}(0,1)$, independent of the walk $\Phi(X(t-1))$
- $$U_N\big(t, x + \Phi(X(t-1))\big) \underset{N}{\longrightarrow} 1 - \exp -e^{-x}$$ a.s. uniformly in $X(t-1)$

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We complete it by

$$\ln N \times \Big\{ U_N\Big(t, x + (t-1)\ln b_N + \Phi(X(0))\Big) - u(x) \Big\} \xrightarrow{\text{law}} u'(x)\big(t\mathcal{S} + t\ln t + tC\big)$$

Perturbations of Gumbel

Perturbations of Gumbel: what is left from these exact computations?

Defining $\epsilon(x)$ by $\boxed{\mathbb{P}(\xi \leq x) = \exp -(1 - \epsilon(x))e^{-x}}$, we assume

$$\lim_{x \to +\infty} \epsilon(x) = 0, \qquad \text{and} \quad \epsilon(x) \in [-\delta^{-1}, 1 - \delta],$$

same tails on the right stronger assumption (domination)

Theorem (CQR 12)

$$U_N(t, x - \Phi(X(t-1))) \longrightarrow u(x) = 1 - e^{-e^{-x}}$$

uniformly in x and on $X(t-2)$ in probability as $N \to \infty$.

Two steps are needed to feel the attraction of the Gumbel

Outline

Front speed for bounded variables

What are the finite-size corrections to the front speed in a case when the distribution of ξ is quite different from the Gumbel law ? Look at bounded case.

Let $b < a$ and $p \in (0,1)$, and assume ξ is integrable and satisfy

$$\mathbb{P}(\xi > a) = \mathbb{P}(\xi \in (b,a)) = 0, \qquad \mathbb{P}(\xi = a) = p, \qquad \mathbb{P}(\xi \in (b-\epsilon, b]) > 0$$

for all $\epsilon > 0$ ($b < a$). Clearly, $v_N \to a$ as $N \to \infty$. Convergence is extremely fast:

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Theorem (CQR 12)

$$v_N = a - (a-b)(1-p)^{N^2}2^N + o\big((1-p)^{N^2}2^N\big), \qquad N \to \infty$$

The leading terms depend only on a few features of the law of ξ: largest value a, probability mass p and the $a-b$ with second largest.
All these involve the top of the support of the distribution, the other details being irrelevant. Such a behavior is expected for pulled fronts.

Front speed for bounded variables

Convergence extremely fast in

$$v_N = a - (a-b)(1-p)^{N^2} 2^N + \dots$$

Parallel to branching random walk with selection, and the Derrida-Brunet correction:
Bérard-Gouéré 2011, Berestycki- Berestycki-Schweinsberg 2011 (and Mueller-Mytnik-Quastel 2011 for stochastic KPP)

$$v_N = v_\infty - \log - \text{corrections}$$

much slower!! They deal with the case

$$2 \times p < 1$$

Our result for bounded ξ correspond to the other case.

First step of proof: front speed for the Bernoulli distribution

Start with Bernoulli case:

$$\mathbb{P}(\xi_{i,j}(t) = 1) = p, \qquad \mathbb{P}(\xi_{i,j}(t) = 0) = q = 1 - p, \qquad p \in (0, 1).$$

Leaders and laggers:

$$Z(t) = \sharp\{j : 1 \leq j \leq N, X_j(t) = 1 + \max\{X_i(t-1); i \leq N\}\}.$$

is a Markov chain on $\{0, 1, \ldots, N\}$ with transitions given by the binomial distributions

$$\mathbb{P}\big(Z(t+1) = \cdot\ |Z(t) = m\big) = \begin{cases} \mathcal{B}\big(N, 1 - q^m\big)(\cdot), & m \geq 1, \\ \mathcal{B}\big(N, 1 - q^N\big)(\cdot), & m = 0. \end{cases}$$

Starting from 0 or from N, the chain has the same law.

Lemma (CQR 12)

$$v_N = 1 - q^{N^2} 2^N + o(q^{N^2} 2^N)$$

Second step: front speed for discrete distribution

We extend the ideas of proof to the case of $\xi \in \{k, k-1, k-2, \ldots\}$: with $p_\ell = \mathbb{P}(\xi = \ell)$, assume

$$p_k \times p_{k-1} > 0$$

and $\mathbb{E}(|\xi_{0,0}|) < \infty$. Then,

$$v_N = k - q_k^{N^2} 2^N + o(q_k^{N^2} 2^N),$$

as $N \to \infty$, where $q_k := 1 - p_k$. (The extension is non trivial).

Sketch of proof of first step

ξ is Bernoulli (p). The speed relates to the invariant measure ν_N of the chain $Z(t) = \sharp\{j : X_j(t) = 1 + \max_i X_i(t-1)\}$, by

$$v_N = 1 - \nu_N(0) = 1 - (E_0 T_0)^{-1} = 1 - (E_N T_0)^{-1}.$$

We prove:

- $P_N(T_0 < T_N) \sim q^{N^2} 2^N$,
- $\lim_{N\to\infty} E_N(T_0 | T_0 < T_N) = 2$,
- $\lim_{N\to\infty} E_N\left(T_N | T_N < T_0\right) = 1$.

We estimate $P_N(T_0 = \ell < T_N)$. For $\ell = 1$,

$$P_N(T_0 = 1 < T_N) = q^{N^2},$$

Contribution of strategies in two steps:

$$\begin{aligned}
P_N(T_0 = 2 < T_N) &= \sum_{k=1}^{N-1} \binom{N}{k} (1 - q^N)^k q^{N(N-k)} \times \binom{N}{0} (1 - q^k)^0 q^{kN} \\
&= q^{N^2} \sum_{k=1}^{N-1} \binom{N}{k} (1 - q^N)^k \\
&= q^{N^2} \left[(2 - q^N)^N - 1 - (1 - q^N)^N\right] \\
&\sim q^{N^2} 2^N.
\end{aligned}$$

And finally (the heart of the problem)

$$P_N(T_0 < T_N, T_0 \geq 3) = o(2^N).$$