Entropic repulsion and metastability in the solid-on-solid (SOS) model

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Plan

- SOS: a random interface model
- Heat Bath Dynamics: Poincaré inequality
- Open problems and conjectures
- SOS with a wall. Entropic repulsion
- Dynamics of entropic repulsion: Metastability
- Exponentially large relaxation times
- Methods

(2+1)Dimensional SOS model

Discrete height: $\varphi = \{\varphi_x,\ x \in \mathbb{Z}^2\}$, with $\varphi_x \in \mathbb{Z}$.

Λ square of side L in $\mathbb{Z}^2$ centered at 0.

0 boundary condition: $\varphi_x = 0$ for all $x \in \mathbb{Z}^2 \setminus \Lambda$.

Gibbs measure: $\beta > 0$

$$\pi(\varphi) = \pi_{\beta,L}(\varphi) = \frac{1}{Z_{\beta,L}} \exp\Big(- \beta \sum_{x \sim y} |\varphi_x - \varphi_y| \Big)$$

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Roughening transition:

Low temperature (large β, rigid phase): localization
$\pi_{\beta,L}(\varphi_0^2) \leqslant C_\beta$ (exponential tails, via Peierls argument)

High temperature (small β, rough phase): delocalization
$\pi_{\beta,L}(\varphi_0^2) \sim \log L$ (difficult ! see Frohlich-Spencer CMP 1981).
[One expects Gaussian fluctuations]

Heat bath dynamics of SOS model

Cont. time π-reversible Markov chain with generator

$$\mathcal{L}f(\varphi) = \sum_{x\in\Lambda} \left[\pi(f \,|\, \varphi_{\Lambda\setminus\{x\}}) - f(\varphi)\right].$$

Dirichlet form $\mathcal{E}(f,f) = \sum_{x\in\Lambda} \pi\,[\mathrm{Var}_x(f)]$

where $\mathrm{Var}_x(f) = \mathrm{Var}_\pi(f \,|\, \varphi_{\Lambda\setminus\{x\}})$.

[Glauber dynamics, Gibbs sampler]

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Poincaré inequality: find $\gamma(L,\pi) > 0$ such that for all $f \in L^2(\pi)$:

$$\mathrm{Var}_\pi(f) \;\leqslant\; \gamma(L,\pi)\,\mathcal{E}(f,f)$$

$\gamma(L,\pi)$ is the *Relaxation Time*, inverse of *Spectral Gap*.

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Open problem: prove that $\gamma(L,\pi)$ is polynomial in L

Remarks on the continuous SOS model

Same problem, but now $\varphi_x \in \mathbb{R}$ and

$$\pi(\varphi) = \frac{1}{Z_{\beta,L}} \exp\Big(-\beta \sum_{x\sim y} |\varphi_x - \varphi_y| \Big)$$

is a log-concave probability on $\mathbb{R}^\Lambda$.
No roughening transition in the continuous model,
surface is always rough in 2D.

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Dynamics:

Langevin diffusion (SDE): $\mathcal{E}(f,f) = \frac{1}{2}\sum_{x\in\Lambda} \pi\left[(\partial_x f)^2\right]$

Poincaré inequality:

$$\mathrm{Var}_\pi(f) \leqslant \gamma(L,\pi)\,\mathcal{E}(f,f)$$

Expected: $\gamma(L,\pi) = O(L^2)$.
As in Gaussian free field case.

$(1+1)$D SOS Model

$$\varphi = \{\varphi_i\ ,\ i = 1,\dots,L\}: \quad \nu_i(d\eta_i) = \frac{e^{-\beta\eta_i}}{Z}\, d\eta_i,$$

$$\pi = \otimes_{i=1}^{L-1} \nu_i\Big(\ \cdot\ \mid \sum_i \eta_i = 0\Big)\,, \quad \eta_i := \varphi_{i+1} - \varphi_i.$$

Continuous heights: $\varphi_i \in \mathbb{R}$, then $\gamma(L,\pi) = O(L^2)$
[Barthe-E.Milman; Barthe-Cordero Erasquin, Barthe-Wolff]

Discrete heights: $\varphi_i \in \mathbb{Z}$, then $\gamma(L,\pi) = O(L^2)$
[Martinelli-Sinclair]

Metropolis chain with ± 1 height updates:

$$\mathcal{E}(f,f) = \frac{1}{2}\sum_x \pi\big[c_x(\nabla_x f)^2\big]$$

C-Martinelli-Toninelli : $\gamma(L,\pi) = O\big(L^2(\log L)^c\big)$
(CMP 2012, approximate motion by mean curvature).

$(2+1)$D SOS (discrete) with a wall: Entropic repulsion

$\varphi_x \in \mathbb{Z}$ and

$$\pi_+(\varphi) = \pi(\varphi \,|\, \varphi_x \;\geqslant\; 0 \;\forall x \in \Lambda)$$

Entropic repulsion heuristics (β large):

- shift heights $h \to h+1$ at energy loss $-4\beta L$ (boundary)
- full downward spikes at x give the gain in entropy $+L^2\, e^{-4\beta h}$.
- surface grows until $4\beta L \approx L^2\, e^{-4\beta h}$ or $h \approx H(L) := \frac{1}{4\beta} \log L$.

Bricmont, El Mellouki, Frhölich '82:
$\frac{1}{|\Lambda|} \sum_{x \in \Lambda} \pi_+[\varphi_x] \in [c_1 \log L, c_2 \log L]$

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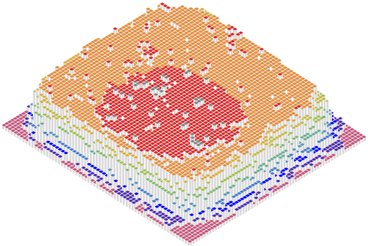
Theorem

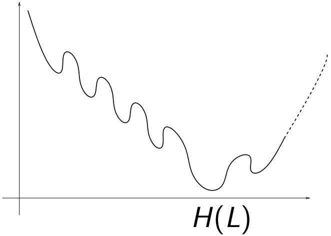
For any $\beta \geqslant \beta_0$, $k \geqslant k_0$:

$$\pi_+\left(\#\left\{x \in \Lambda :\; \varphi_x \notin [H(L)-k, H(L)+k]\right\} > e^{-2\beta k} L^2\right) \leqslant e^{-cL}.$$

Metastability

A deeper analysis of the free energy landscape reveals metastable behavior:





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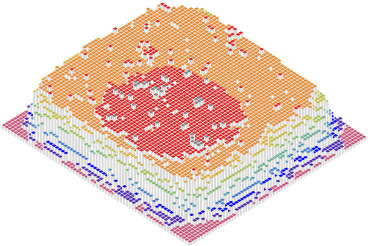

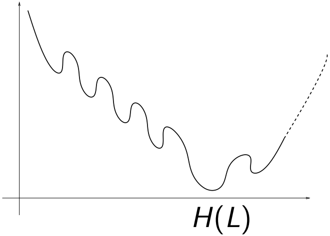

Theorem

Start heat bath dynamics at $\varphi \equiv 0$.
For $a \in (0,1]$, *let* $\tau_a = \min\{t > 0 : \varphi(t) \in \Omega_a\}$ *where* $\Omega_a = \left\{\varphi : \#\{x : \varphi_x \geq aH(L)\} > 0.9\, L^2\right\}$.
Then $\lim_{L\to\infty} \pi_+(\Omega_a) = 1$ *and yet*

$$\lim_{L\to\infty} \mathbb{P}\left(e^{cL^a} \leq \tau_a \leq e^{(1/c)L^a}\right) = 1\,.$$

Mixing time bounds

SOS model with wall and ceiling

$$\pi_{+,b}(\varphi) = \pi(\varphi \,|\, 0 \leqslant \varphi_x \leqslant L\,,\ \forall x \in \Lambda)$$

Heat bath is then a Markov chain with finite state space.

$$T_{\mathrm{mix}}(L) = \inf\left\{t > 0 : \max_{\varphi} \|p_t(\varphi,\cdot) - \pi_{+,b}\|_{\mathrm{TV}} \le \tfrac{1}{2}\right\}$$

From standard bounds:

$$\gamma(L, \pi_{+,b}) \leqslant c\, T_{\mathrm{mix}}(L) \leqslant c' L^3 \gamma(L, \pi_{+,b}).$$

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Theorem

For any $\beta \geqslant \beta_0$, $\exists c > 0$:

$$e^{c\,L} \leq T_{\mathrm{mix}}(L) \leq e^{(1/c)\,L}.$$

Spectral gap bounds

Without ceiling (π_+). Suitable recursive analysis shows that

$$\gamma(L, \pi_+) \approx \gamma(L, \pi_{+,b})$$

Then from previous theorem, one has

$$e^{c\,L} \leq \gamma(L, \pi_+) \leq e^{(1/c)\,L}.$$

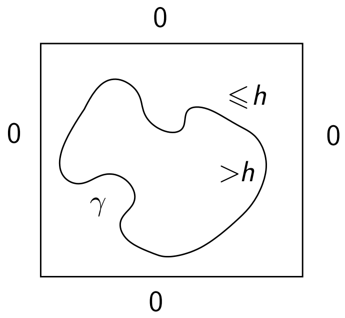
Without wall (π). Expect polynomial $\gamma(L, \pi)$. We prove

Theorem

$$\gamma(L, \pi) \leq e^{L^{5/6}}.$$

Methods 1

Equilibrium estimates for $\pi, \pi_+, \pi_{+,b}$: Monotonicity, FKG inequalities, Peierls type estimates (contour estimates)



$$\pi_+(\gamma \text{ is an } h-\text{contour}) \;\leqslant\; \exp\Big\{ -\beta|\gamma| + c\,\mathrm{Area}(\gamma) e^{-4\beta h} \Big\}.$$

Methods 2

Metastability analysis and lower bounds on mixing times:
Refined equilibrium bounds to quantify bottlenecks

Main idea: Fix $h = aH(L) = \frac{a}{4\beta}\log L$.
Restricted ensemble $\pi_A = \pi_+(\cdot \mid A)$, where A is the event that all h contours γ have Area$(\gamma) \leqslant \delta L^{2a}$, δ small.

Then in π_A:

1) all h contours have area less than $(\log L)^2$ w.h.p.

2) $\pi_A(\partial A) \leqslant e^{-cL^a}$

3) π_A (large density of heights at least $h+1$) $\leqslant e^{-cL^a}$

This establishes bottleneck:
$\tau_a \geqslant e^{cL^a}$ w.h.p. when started from $\varphi \equiv 0$.

Methods 3

Upper bounds on mixing times: coupling arguments, with monotonicity and a *standard canonical paths* technique yield the upper bound

$$T_{\rm mix}(L) \;\leqslant\; e^{c\,L\log L}$$

To obtain $T_{\rm mix}(L) \;\leqslant\; e^{c\,L}$ much more work is needed.

Main steps:

- Improved *canonical paths* argument: define reduced space $G \subset \Omega$ such that on G canonical paths gives $T_{\rm mix}(L;G) \;\leqslant\; e^{c\,L}$.
- If T_G is time needed to enter G with probab. α, then $T_{\rm mix}(L) \;\leqslant\; c(\alpha)(T_G^2 + T_{\rm mix}(L,G))$.
- Show that uniformly in initial condition, the process enters G within time $e^{cL} \sim T_G$ with probab. at least $\alpha \sim 1/2$.
- *Cluster expansion* tools to have fine control of the statistics of SOS contours.