Metastability & interface motion in disordered media

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Joint work with

B. Graham, A. Teixeira, M. Wouts

IPS conference

Outline

Metastability for the dilute Ising model

- Ising Model
- Glauber dynamics and metastability
- Random interactions and catalyst effect

Interface motion in random media

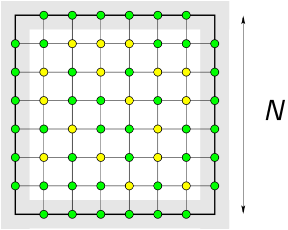
- Zero temperature phase transition
- Positive velocity & renormalization procedure

Domain $\Lambda \subset \mathbb{Z}^d$

Configurations : $\sigma_\Lambda = \{\sigma_i\}_{i \in \Lambda} \in \{-1, 1\}^\Lambda$

Nearest neighbor interactions and boundary conditions

$$H^+(\sigma_\Lambda) = -\sum_{\substack{i \sim j \\ i,j \in \Lambda}} \sigma_i \sigma_j - \sum_{\substack{i \sim j \\ i \in \Lambda, j \notin \Lambda}} \sigma_i$$



Gibbs measure

$$\mu^+_{\beta,\Lambda}(\sigma_\Lambda) = \frac{1}{Z^+_{\beta,\Lambda}} \exp\left(-\beta H^+(\sigma_\Lambda)\right)$$

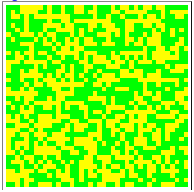
For $\Lambda_N = \{-N, N\}^d$ define $\boxed{\mu^+_{\beta,N} = \mu^+_{\beta,\Lambda_N}}$

Thermodynamic limit $\lim_{N\to\infty} \mu^+_{\beta,N} = \mu^+_\beta$

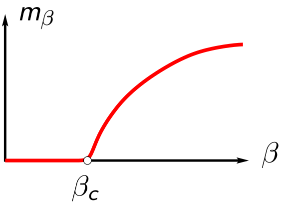
Phase transition

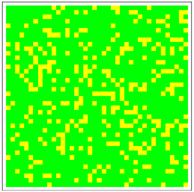
Thermodynamic limit $\lim_{N\to\infty} \mu^+_{\beta,N} = \mu^+_\beta$

Magnetization $m_\beta = \mathbb{E}_{\mu^+_\beta}(\sigma_0)$



$\beta < \beta_c$





$\beta > \beta_c$

There is a critical value β_c such that $\beta > \beta_c \;\Leftrightarrow\; m_\beta > 0$

Influence of the boundary

$$\beta < \beta_c \;\Rightarrow\; \mu^+_\beta = \mu^-_\beta$$
$$\beta > \beta_c \;\Rightarrow\; \mu^+_\beta \neq \mu^-_\beta$$

Magnetic Field h

Interaction and Magnetic Field :

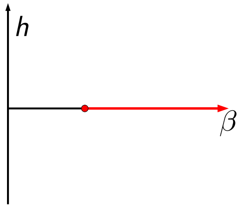
$$H^h(\sigma_\Lambda) = -\sum_{\substack{i\sim j\\ i,j\in\Lambda}} \sigma_i\sigma_j \; - h\sum_{i\in\Lambda}\sigma_i$$

Gibbs measure $\qquad \mu^h_{\beta,\Lambda}(\sigma_\Lambda) = \frac{1}{Z^h_{\beta,\Lambda}} \exp\big(-\beta H^h(\sigma_\Lambda)\big)$

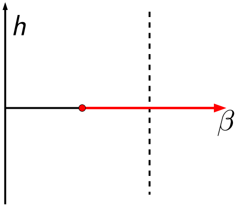
$h \neq 0$

No influence of the boundary $\qquad \beta > 0 \;\Rightarrow\; \mu_\beta^{h,+} = \mu_\beta^{h,-}$

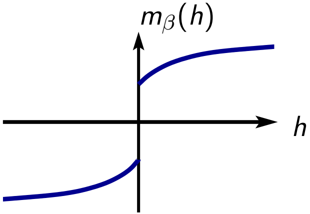
$h \neq 0$: unique measure μ^h_β on $\mathbb{Z}^d$



Magnetic Field h



$$m_\beta(h) = \mathbb{E}_{\mu_\beta^h}(\sigma_0)$$



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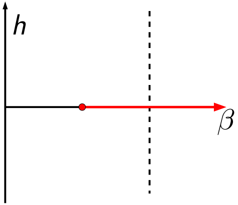

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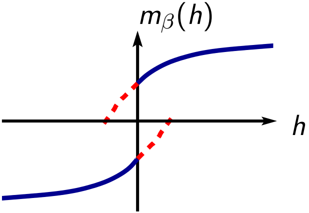
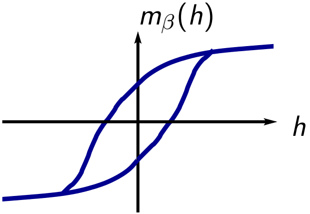
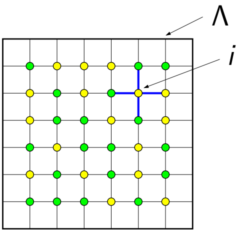


Experimentally: Hysteresis



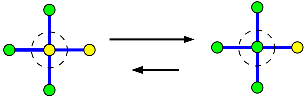
Glauber dynamics : Markov Chain

Glauber dynamics is reversible for the Gibbs measure $\mu_{\beta,\Lambda}^{h}$



1. Choose randomly i in Λ
2. Flip $\sigma_i \to -\sigma_i$ depending on
   - ➪ nearest neighbor spins
   - ➪ the magnetic field

$$\text{Rate} = \exp\Big(-\beta\sigma_i\big(\textstyle\sum_{j\sim i}\sigma_j + h\big)\Big)$$



$h \simeq 0$ and $\beta \gg 1$

The dynamics tends to align a spin with its neighbors and the magnetic field.

Metastability

Fix $\beta > \beta_c$ and $h > 0$ then $\quad m_\beta(h) = \mathbb{E}_{\mu_\beta^h}(\sigma_0) > m_\beta > 0$

μ_β^h is the unique invariant measure for the Glauber dynamics on $\mathbb{Z}^d$

Question

Relaxation time of the dynamics starting from $\ominus = \{\sigma_i = -1\}_{i \in \mathbb{Z}^d}$

Metastability

Fix $\beta > \beta_c$ and $h > 0$ then $m_\beta(h) = \mathbb{E}_{\mu_\beta^h}(\sigma_0) > m_\beta > 0$

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Relaxation time of the dynamics starting from $\ominus = \{\sigma_i = -1\}_{i \in \mathbb{Z}^d}$

Theorem [Schonmann, Shlosman]

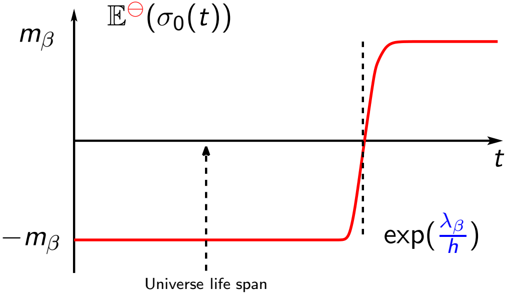
When $d = 2$, there exists $\lambda_\beta > 0$ such that

$$t \ll \exp\left(\frac{\lambda_\beta}{h}\right), \qquad \mathbb{E}^{\ominus}(\sigma_0(t)) = -m_\beta + o(h)$$

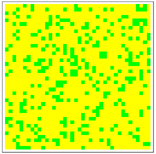
$$t \gg \exp\left(\frac{\lambda_\beta}{h}\right), \qquad \mathbb{E}^{\ominus}(\sigma_0(t)) = m_\beta + o(h)$$

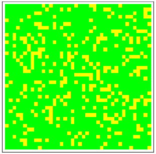
Metastability

Choose $\beta > \beta_c$ and $h \approx 0$



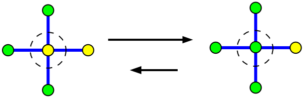
The minus phase is metastable for small h





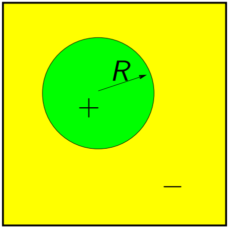
Nucleation

$h \approx 0$



Forming a droplet of $+$ of radius R

- Surface cost $\approx \tau_\beta R$
- Bulk gain $\approx h m_\beta R^2$



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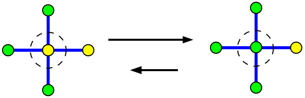

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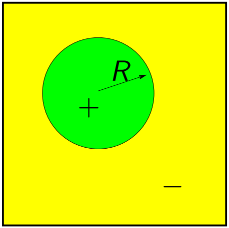
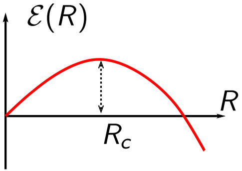


Minimize the droplet energy

$$\mathcal{E}(R) = \tau_\beta R - h m_\beta R^2$$

Energy barrier at $R_c = \frac{\tau_\beta}{2m_\beta}\frac{1}{h}$



Nucleation and Droplet growth

Nucleation time $\approx \exp\left(\mathcal{E}(R_c)\right) = \exp\left(\frac{\tau_\beta^2}{4m_\beta}\frac{1}{h}\right)$

[Olivieri, Vares]
[Cerf, Ben Arous], [Bovier, Eckhoff, Gayrard, Klein]
[Gaudillière, Den Hollander, Nardi, Olivieri, Scoppola]
[Beltran, Landim] ....

Nucleation and Droplet growth

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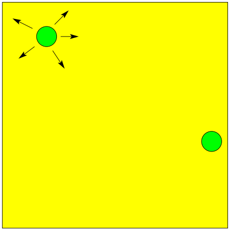
[Beltran, Landim] ....

Nucleation anywhere in space and then droplet growth

➪ Corrections on the relaxation time

$$\text{Relaxation time} \simeq \exp\left(\frac{1}{d+1}\ \frac{\tau_\beta^2}{4m_\beta}\frac{1}{h}\right)$$

[Dehghanpour, Schonmann]
[Schonmann, Shlosman]



Random interactions – Modeling Alloys

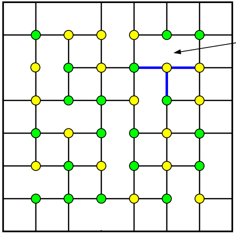
Edges in $\mathbb{Z}^d$ are removed independently with probability $1 - p$

$$i \sim j, \qquad \mathbb{Q}\big(J_{(i,j)} = 1\big) = 1 - \mathbb{Q}\big(J_{(i,j)} = 0\big) = p$$

Configurations: $\{\sigma_i\}_{i\in\Lambda} \in \{-1,1\}^{\Lambda}$

Nearest neighbor interactions

$$H^J(\sigma_\Lambda) = -\sum_{\substack{i\sim j \\ i,j\in\Lambda}} J_{(i,j)}\sigma_i\sigma_j$$



No interaction

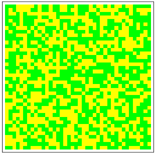
Quenched Gibbs measure

$$\mu^J_{\beta,\Lambda}(\sigma_\Lambda) = \frac{1}{Z^J_{\beta,\Lambda}} \exp\Big(-\beta H^J(\sigma_\Lambda)\Big)$$

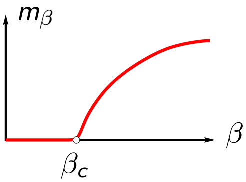
Phase transition

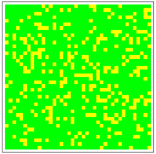
Magnetization $\quad m_\beta = \mathbb{Q}\left(\mathbb{E}_{\mu_\beta^{J,+}}(\sigma_0)\right) \qquad \lim_N \mu_{\beta,N}^{J,+} = \mu_\beta^{J,+}$

Fix $p > p_c > 0$



$\beta < \beta_c$





$\beta > \beta_c$

There is a critical value $\beta_c = \beta_c(p)$ such that $\beta > \beta_c \;\Leftrightarrow\; m_\beta > 0$

Influence of the boundary

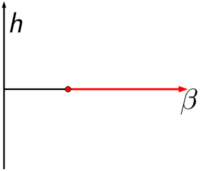
$$\beta < \beta_c \;\Rightarrow\; \mu_\beta^{J,+} = \mu_\beta^{J,-}, \quad J \text{ a.s.}$$
$$\beta > \beta_c \;\Rightarrow\; \mu_\beta^{J,+} \neq \mu_\beta^{J,-}, \quad J \text{ a.s.}$$

Magnetic Field h

Hamiltonian: $H^{J,h}(\sigma_\Lambda) = -\sum_{\substack{i\sim j \\ i,j\in\Lambda}} J_{(i,j)}\sigma_i\sigma_j \; - h \; \sum_{i\in\Lambda}\sigma_i$

Gibbs measure $\mu^{J,h}_{\beta,\Lambda}(\sigma_\Lambda) = \frac{1}{Z^{J,h}_{\beta,\Lambda}} \exp\left(-\beta H^{J,h}(\sigma_\Lambda)\right)$

$h \neq 0$: unique measure $\mu^{J,h}_\beta$ on $\mathbb{Z}^d$



Question

Impact of the disorder on the dynamics ?

Relaxation time of the dynamics starting from $\ominus = \{\sigma_i = -1\}_{i\in\mathbb{Z}^d}$

Previous results

[Guionnet, Zegarlinski], [Cesi, Maes, Martinelli]

Slowdown of the dynamics in the uniqueness regime

$$\mathbb{Q}\left(\mathbb{E}_{\mu_\beta^{J,+}}(\sigma_0)\right) = 0 \qquad (\text{with } h = 0)$$

- No disorder : exponential relaxation to equilibrium
- Disorder (edge dilution) then for some range of (β, p) relaxation like $\exp(-(\log t)^{\frac{d}{d-1}})$

[Fontes, Mathieu, Picco], [Bianchi, Bovier, Ioffe]

Metastability for the Curie Weiss random field Ising model

[Wouts] : Spectral gap & relaxation in a pure phase

Faster relaxation to equilibrium

$\mu_\beta^{J,h}$ unique invariant measure for the Glauber dynamics on $\mathbb{Z}^d$

Theorem [B, Graham, Wouts]

Fix $d \geq 2$ and $\beta \succ \beta_c(p)$. Then there exists $\lambda_\beta(p) > 0$ such that

$$t \gg \exp\left(\frac{\lambda_\beta(p)}{h^{d-1}}\right),$$

$$\mathbb{Q}\left(\mathbb{E}^{J,\ominus}(\sigma_0(t)) = \mathbb{E}_{\mu_\beta^{J,h}}(\sigma_0) + o(h)\right) = 1 - o(h)$$

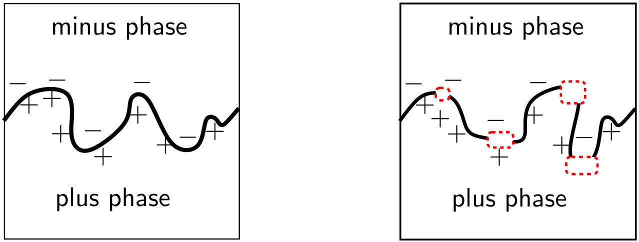
Disorder facilitates the relaxation

$$\forall p < 1, \qquad \lim_{\beta\to\infty} \frac{\lambda_\beta(p)}{\lambda_\beta(\text{quenched})} = 0$$

Reminiscent of catalysts in chemical reactions.

Catalyst effect

The disorder lowers the phase coexistence cost



Remark. In $d = 2$:

Interface with random interactions $\simeq$ Polymer in random environment

[Huse, Henley]

Catalyst effect

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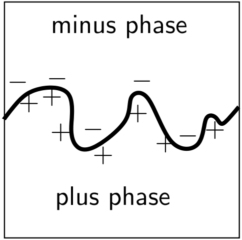
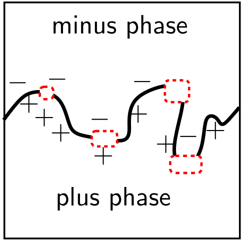
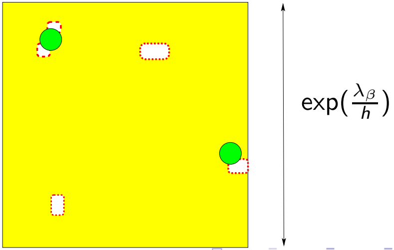


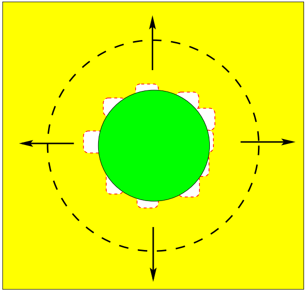


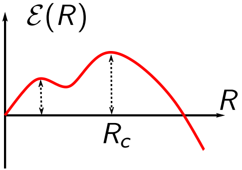
Atypical regions with high dilution act as catalysts and facilitate the nucleation



Slowdown by the disorder

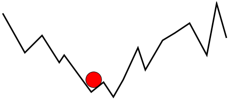
Slowdown of the droplet growth by rare traps with high disorder





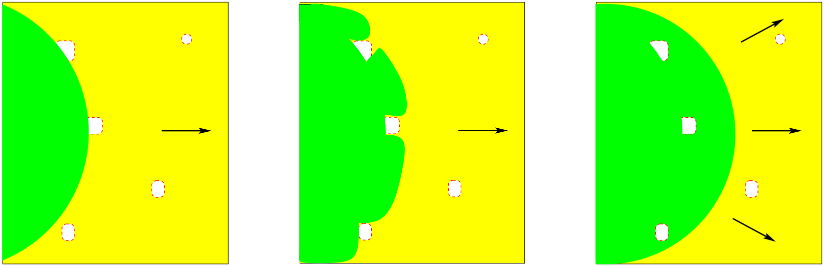
Energy landscape with disorder

Similar mechanism for a Random Walk in Random Environnement



Later stage of droplet growth

For very large droplets the analogy with Random Walk in Random Environnement is no longer valid.

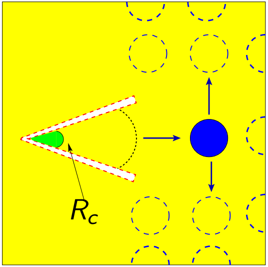


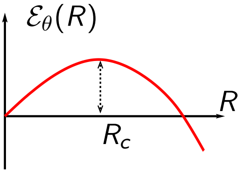
One can derive a (crude) lower bound on the growth velocity

Open question

Understanding interface velocity ⇔ Impact of disorder at all scales

Cone catalysts





Energy landscape in a cone of angle θ
Energy barrier is of order θ^d

Two step growth

- Nucleation in a cone catalyst with angle θ (atypical event)
- Invasion by large droplets (super-critical percolation).

Mathematical tools

Key issue: Phase coexistence with disorder & Renormalization

[Schonmann, Shlosman] used a two-dimensional approach devised by [Dobrushin, Kotecky, Shlosman, Pfister, Ioffe, Velenik]

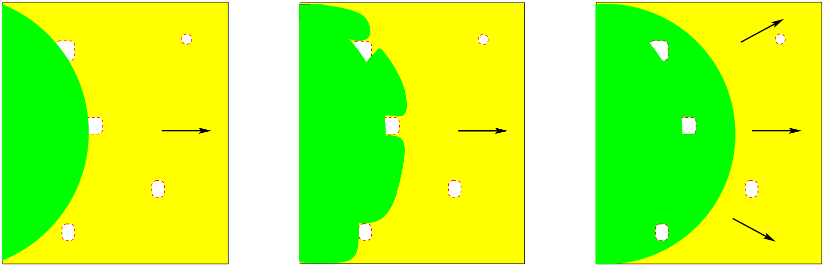
We rely on the $\mathbb{L}^1$-approach introduced by [Presutti, Cassandro, Alberti, Belletini, Cerf, Pisztora, B., Ioffe, Velenik]

The $\mathbb{L}^1$-approach was extended to disordered systems by [Wouts]. This method allows us to control deviations of the surface tension.

Byproduct

Generalization in $d \geq 3$ of the upper bound on the relaxation time derived in [Schonmann, Shlosman]

Later stage of droplet growth



Question

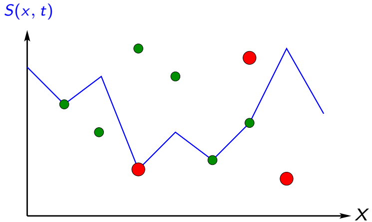
Understanding interface velocity $\Leftrightarrow$ Impact of disorder at all scales

Effective interface model

Interface heights :
$x \in \mathbb{Z}, t \in \mathbb{Z}^+, \quad S(x,t) \in \mathbb{Z}^+$

Disorder :
$x \in \mathbb{Z}, y \in \mathbb{Z}^+, \quad \eta(x,y) \in \mathbb{R}$

i.i.d variables and $\mathbb{E}(\eta) = f \geq 0$



Random force with positive mean

$$" \ \partial_t S(x,t) = \Delta S(x,t) + \eta\big(x, S(x,t)\big) \ "$$

Effective interface model

Interface heights :

$x \in \mathbb{Z}, t \in \mathbb{Z}^+, \quad S(x,t) \in \mathbb{Z}^+$

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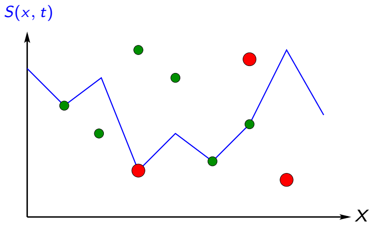
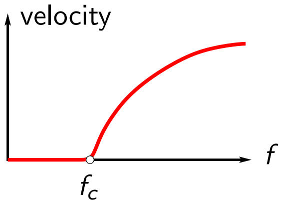


Zero temperature dynamics

- Initial data : $\quad S(x,0) = 0$
- $S(x,t+1) = S(x,t) + 1$ if

$$S(x+1,t) + S(x-1,t) - 2S(x,t) + \eta\big(x, S(x,t)\big) > 0$$

Zero temperature : phase transition



$f < f_c$: the interface is blocked

$f > f_c$: positive velocity

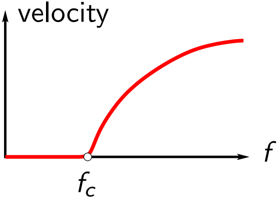
Physics :

[Koplik, Levine], [Narayan, Fisher], [Leschhorn], [Vannimenus, Derrida], [Schütze, Nattermann], [Le Doussal, Wiese, Chauve] [Giamarchi, Kolton, Krauth, Rosso]

Open question

Critical exponents

Zero temperature : phase transition



$f < f_c$: the interface is blocked

$f > f_c$: positive velocity

Mathematics :

$f \simeq 0$:
[Dirr, Dondl, Grimmett, Holroyd, Scheutzow], [Dirr, Dondl, Scheutzow]

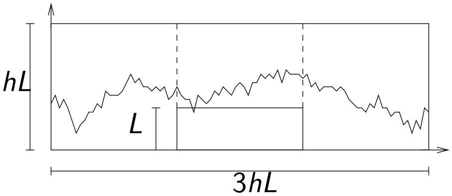
$f \gg 1$:
[Coville, Dirr, Luckhaus], [Dondl, Scheutzow]

Open question

Could the interface move with zero velocity ?

Criterion for positive velocity

Let $h > 1$. Define the set of blocked interfaces

$\mathcal{A}^{h,L} = \Big\{ \eta;$  $\Big\}$

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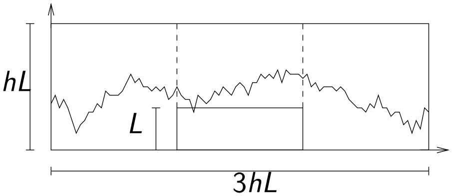
$\mathcal{A}^{h,L} = \Big\{ \eta;$  $\Big\}$

Criterion

Suppose there is $h > 1$, $\rho > 0$ such for L large enough

$$\mathbb{P}\big(\mathcal{A}^{h,L}\big) \leq \frac{1}{L^{\rho}}$$

Criterion for positive velocity

Theorem. [B, Teixeira]

Suppose that the criterion holds then there is $c > 0$ such that

$$\liminf_{t\to\infty} \frac{1}{t} S(0,t) \geq c$$

Perturbative Regime :

Suppose that $\{\eta(x,y)\}$ are i.i.d Gaussian variables with

- Mean : $\mathbb{E}(\eta) = f$
- Variance : $\mathbb{E}(\eta^2) - \mathbb{E}(\eta)^2 = \sigma$

If f is large enough then the criterion holds.

[Coville, Dirr, Luckhaus], [Dondl, Scheutzow]

Criterion for positive velocity

Theorem. [B, Teixeira]

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Analogy with percolation :

- $p < p_c$: Exponential decay of $\mathbb{P}(O \leftrightarrow x)$ when $x \to \infty$ [Aizenman, Barsky], [Menshikov]
- $p > p_c$: Slab percolation [Aizenman, Chayes, Chayes, Russo], [Grimmett, Marstrand]

Conclusion

- Glauber dynamics and metastability
- Random interactions and catalyst effect
- Phase transition for interfaces in random media
- Criterion for positive speed

Open problems

- Metastability : Lower bound on the relaxation time
- Validity of the criterion up to f_c