Front progression in the East model

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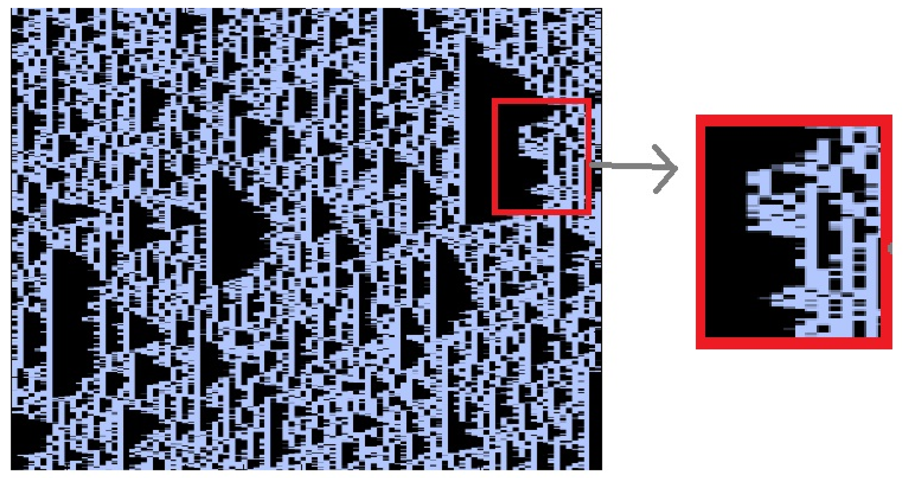
Model and motivations



A simulation of the East model at equilibrium with density 1/2 ; the ones are in black.

For $p \in (0,1)$, the East model with density p is defined on $\mathbb{Z}$ by the generator

$$\mathcal{L}f(\omega) = \sum_{x \in \mathbb{Z}} (1 - \omega_{x+1})(p(1-\omega_x) + (1-p)\omega_x)\left[f(\omega^x) - f(\omega)\right]$$

It has equilibrium (and in fact reversible) measure μ the product Bernoulli measure of density p on $\{0,1\}^{\mathbb{Z}}$.

It can also be constructed by attaching a Poisson process clock to every site in $\mathbb{Z}$. Every time a clock rings, the site is refreshed to a Bernoulli variable of parameter p, but only if its right neighbour is empty.

This constraint creates the "bubbles" one can see on the picture. A natural question is : how do these bubbles appear and disappear?

We begin the investigation of this matter by considering the East model with initial configuration empty on the negative sites. We then ask the following questions :

- How does the position of the left-most zero (*the front*) behave?
- What does the *configuration seen from the front* look like?

Note : Since the dynamics is not monotonic, specific arguments have to be devised to address these questions. For instance, the sub-additivity argument used in [Lig85] relies heavily on attractiveness and cannot transfer to our case.

Main results

• Law of large numbers for the position of the front

Theorem 1 *There exists $v < 0$ such that if X_t is the position of the front at time t*

$$\frac{X_t}{t} \xrightarrow[t\to\infty]{} v$$

• Ergodicity of the system seen from the front

Theorem 2 *The process of the configurations seen from the front admits a unique invariant measure.*

References

[AD02] David Aldous and Persi Diaconis. The asymmetric one-dimensional constrained Ising model: rigorous results. *J. Statist. Phys.*, 107(5-6):945–975, 2002.

[CMST10] N. Cancrini, F. Martinelli, R. Schonmann, and C. Toninelli. Facilitated oriented spin models: some non equilibrium results. *J. Stat. Phys.*, 138(6):1109–1123, 2010.

[KS01] Sergei Kuksin and Armen Shirikyan. A coupling approach to randomly forced nonlinear PDE's. I. *Comm. Math. Phys.*, 221(2):351–366, 2001.

[Lig85] Thomas M. Liggett. *Interacting particle systems*, volume 276 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

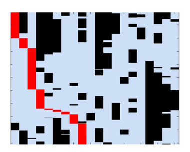
Key ingredients

Distinguished zero

The distinguished zero is a tool introduced in [AD02] that exploits the orientation of the model.

Choose a zero in the initial configuration and make it distinguished. It remains the distinguished zero up to the first time its clock rings while its right neighbour is also a zero. It then jumps one site to the right. We then iterate the process. Notice that by construction there is always a zero at the position of the distinguished zero.

Moreover, the distinguished zero leaves equilibrium behind it : every time it jumps to the right, the site it occupied previously is instantaneously put to equilibrium (a Bernoulli variable independant of all the rest). Then, until the distinguished zero moves again, we have an East dynamics with zero boundary condition, for which the equilibrium product measure is stationary.



An example of the trajectory of a distinguished zero (in red)

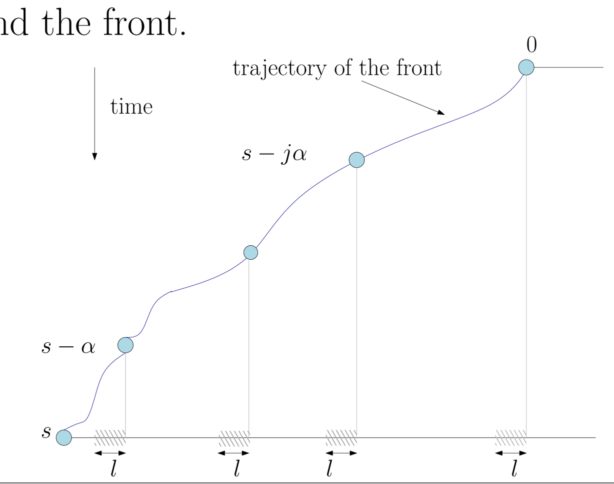
One interest of this object is that it acts as a moving zero boundary condition, thus enabling relaxation on its left :

Proposition 1 *Let $\omega \in \{0,1\}^{\mathbb{Z}}$ such that $\omega_z = 0$ for some $z > 0$, and f a bounded function with support in $\mathbb{N}$. Let $(\xi(t))_{t\geq 0}$ be the trajectory of the distinguished zero starting at z. Then*

$$\left|\mathbb{E}_\omega\left[f(\omega(t))\right] - \mathbb{E}_\omega\left[\mu_{\{0\}}(f)(\omega(t))\right]\right| \leq \sqrt{2}\|f\|_\infty \left(\frac{1}{p \wedge q}\right)^z e^{-t\text{gap}}$$

This result is more precise than the one stated in [CMST10], in that it applies to function with infinite support.

We use this repeatedly, distinguishing the front at different times, to prove that there are "a lot of zeros" behind the front.



For an appropriate choice of parameters, with high probability there is a zero in each of the shaded boxes : they have relaxed to equilibrium thanks to the zero that was at the front at times $s-\alpha$, $s-2\alpha$, etc.

Decorrelation behind the front

We focus here on what a configuration as seen from the front looks like. Its distribution is not the equilibrium one (product of Bernoulli measure). However, we show that under appropriate assumptions, the total variation distance between the configuration at distance L behind the front and the equilibrium distribution is exponentially small in L.

Theorem 3 *Let L, M be two natural integers. For ω with a left-most zero in 0 (resp. π a distribution on the space of such configurations), $t > 0$, we denote by $\nu^\omega_{t,L,M}$ (resp. $\nu^\pi_{t,L,M}$) the distribution of $\omega(t)_{|[X_t+L+1, X_t+L+M]}$ when $\omega(0) = \omega$ (resp. $\omega(0) \sim \pi$). Then there exist constants $\epsilon > 0$, $K < \infty$ depending only on p such that :*

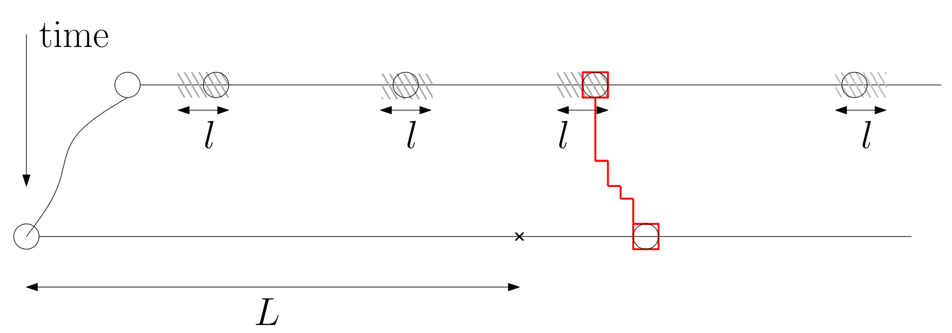
1. If t is large enough (depending on $L+M$), then

$$\|\nu^\omega_{t,L,M} - \mu_{|[1,M]}\|_{TV} \leq K e^{-\epsilon L} \tag{1}$$

2.

$$\left\|\mu - \nu^\mu_{t,L,\infty}\right\|_{TV} \leq K e^{-\epsilon L} \tag{2}$$

To prove this, we use the fact that we established before, that there are several zeros behind the front. With high probability there is one at an appropriate position (in an appropriate box) that we can distinguish and which will bring equilibrium at $X_t + L + 1$, using proposition 1. Then we iterate at $X_t + L + 2, X_t + L + 3, \ldots$. For an appropriate choice of parameters, the error terms are in $e^{-\epsilon(L+k)}$ and sum up to a term in $e^{-\epsilon L}$.



This result is almost enough to prove the first part of theorem 1, using a sub-additivity argument. But we can also deduce it from the second part of theorem 1.

Coupling

To show part 2 of theorem 1, we need to show that we can couple the processes seen from the front starting from any two different configurations. The previous theorem tells us that we can do it far from the front, where the system tends to forget that there is a front at all. The problem is now to get closer to the front.

Theorem 4 *Let $t > 0$, ω, σ initial configurations. There exists $L = L(t) \in \mathbb{N}^*$, and a coupling (ω_t, σ_t) between $\omega(t)$ and $\sigma(t)$ seen from their respective fronts, such that $L(t) \xrightarrow[t\to\infty]{} +\infty$ and the convergence*

$$\mathcal{P}\left((\omega_t)_{[1,L]} = (\sigma_t)_{[1,L]}\right) \xrightarrow[t\to\infty]{} 1 \tag{3}$$

occurs uniformly in ω, σ.

This gives the unicity of the invariant measure for the process seen from the front.

Inspired by [KS01], our strategy consists in trying until it works.

Basically, we wait long enough for the configurations "far from the front" to couple, using theorem 3. Once this is done, there is a small, but positive probability that the parts of the configurations "near the front" will agree after some time (if an appropriate sequence of clock rings and coin flips occurs). It is enough for us that one out of many attempts be successful. What we need to be careful about is that the failure of a given attempt might compromise the following ones.

However, keeping track of "enough zeros behind the front" throughout the attempts ensures that the coupling far from the front remains successful enough for our purpose. In addition, we can choose the events on which the configurations near the front agree after a successful coupling far from the front to be independent. These two facts guarantee that the probability of success remains high enough throughout the attempts, hence that with high probability one of them will be successful.